The following assignment is due on October 25th at 9:30 am. Please show all your work on a separate piece of paper.

1. Let F_n be the nth Fibonacci number. That is $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Write out the 6 terms of the following sequences.

   (a) $u_n = 5F_n$
   The first 6 Fibonacci numbers are 1,1,2,3,5,8. To find u_n we multiply each term by 5 to get 5,5,10,15,25,40.

   (b) $u_n = 2F_{n+1} - F_n$
   Using the Fibonacci numbers we get

$$\begin{aligned} u_1 &= 2F_2 - F_1 = 2(1) - 1 = 1 \\ u_2 &= 2F_3 - F_2 = 2(2) - 1 = 3 \\ u_3 &= 2F_4 - F_3 = 2(3) - 2 = 4 \\ u_4 &= 2F_5 - F_4 = 2(5) - 3 = 7 \\ u_5 &= 2F_6 - F_5 = 2(8) - 5 = 11 \\ u_6 &= 2F_7 - F_6 = 2(13) - 8 = 18 \end{aligned}$$

   Show that both the sequences above are approximately geometric when n is large and evaluate the approximate growth factor.
   In each case we find the ratio of neighbouring terms and see if this ratio becomes constant for larger n. For part (a) the ratios are 5/5=1, 10/5=2, 15/10=1.5, 25/15=1.667..., 40/25=1.6... etc. The ratios converge to 1.618... . For part (b) the ratios are 3/1=3, 4/3=1.333, 7/4=1.75, 11/7=1.57.., 18/11=1.63... etc. Again the ratios converge to 1.618 ... . In fact, one can show that all sequences of the form $u_n = aF_{n+1} + bF_n$ are approximately geometric with common ratio equal to the golden ratio $\phi = 1.618...$ for large n.

2. Given that the golden ratio ϕ satisfies the equation $\phi^2 = 1 + \phi$ Show, by multiplying both sides of this equation by ϕ that:

   (a) $\phi^3 = 2\phi + 1$
   $\phi^3 = \phi^2 \cdot \phi = (1 + \phi)\phi = \phi + \phi^2 = \phi + (1 + \phi) = 2\phi + 1$

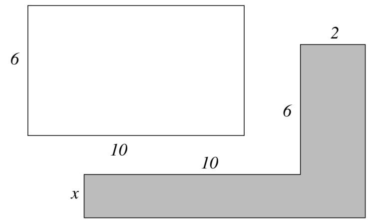
   (b) $\phi^4 = 3\phi + 2$
   $\phi^4 = \phi^3 \cdot \phi = (2\phi + 1)\phi = 2\phi^2 + \phi = 2(1 + \phi) + \phi = 3\phi + 2$

   (c) $\phi^5 = 5\phi + 3$
   $\phi^5 = \phi^4 \cdot \phi = (3\phi + 2)\phi = 3\phi^2 + 2\phi = 3(1 + \phi) + 2\phi = 5\phi + 3$

   Write down an expression for ϕ^n in terms of Fibonacci numbers.
   Based on the above pattern we can see that $\phi^n = \phi F_{n+1} + F_n$.

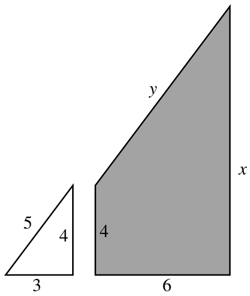
3. Find the value of x so that the shaded area is a gnomon to the rectangle.
Using the fact that the rectangles must be geometrically similar we have:

$$\frac{6+x}{6} = \frac{10+2}{10} = 1.2$$
$$\Rightarrow \quad 6+x = 1.2(6) = 7.2 \;\Rightarrow\; x = 1.2$$

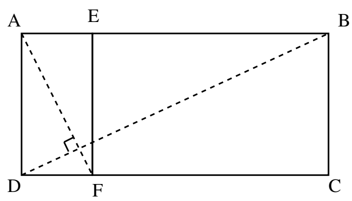


4. Find the values of x and y so that the shaded area is a gnomon to the white triangle.
Using the fact that the triangles must be geometrically similar we have:

$$\frac{x}{4} = \frac{9}{3} = 3 \;\Rightarrow\; x = 12$$
$$\frac{5+y}{5} = \frac{9}{3} = 3 \;\Rightarrow\; 5+y = 15 \;\Rightarrow y = 10$$



5. Let ABCD be an arbitrary rectangle as shown in the figure below. Let AF be perpendicular to the diagonal BD and EF perpendicular to AB. Show that the rectangle BCEF is a gnomon to the rectangle ADFE.



For rectangle BCEF to be gnomon to the rectangle ADFE we must show that rectangle ABCD is geometrically similar to rectangle ADFE. We can do this by observing that triangles ADF and ADB are geometrically similar because their interior angles are equal (They are right angled triangles and angle DAF = angle ABD so angle ADB = angle AFD). Since the triangles are similar the ratios of their sides are equal. Hence $\frac{\text{AB}}{\text{AD}} = \frac{\text{AD}}{\text{DF}}$. This is exactly the result that shows that rectangle ABCD is geometrically similar to rectangle ADFE, so we are finished.