Answer the following questions on a separate sheet of paper and turn in your responses by 9:30 am on Tuesday November 1st.

1. Answer the following questions based on the parastichies patterns you generated with your group in this week's workshop.

   (a) Which divergence angles result in parastichies that are straight lines? How many lines are there in each case. Explain, mathematically, why they line up in a straight line. Explain how the divergence angle relates to the number of straight lines that emerge.
   Please see the worksheet solutions (attached) for graphs of the parastichies. The divergence angles which have straight line parastichies are 45^o with 8, 90^o with 4, 120^o with 3, 135^o with 8 and 150^o with 12 (In the last case the straight lines may not be apparent until you place at least 36 dots.). These parastichies are all straight line parastichies because $\frac{45}{360} = \frac{1}{8}$ so after 8 dots have been placed the next dot is placed a full 360^o rotation from the first – making it inline with the first. Similarly $\frac{90}{360} = \frac{1}{4}$ so there are 4 straight parastichies, $\frac{120}{360} = \frac{1}{3}$ so there are 3 straight parastichies, $\frac{135}{360} = \frac{3}{8}$ so there are 8 parastichies (after 8 dots are placed, the next corresponds to a rotation of 3×360^o and so lines up with the the first dot), and $\frac{150}{360} = \frac{5}{12}$ so there are 12 parastichies. Note in the last case

   $$\frac{5}{12} = \frac{1}{\frac{12}{5}} = \frac{1}{2+\frac{2}{5}} = \frac{1}{2+\frac{1}{\frac{5}{2}}} = \frac{1}{2+\frac{1}{2+\frac{1}{2}}} .$$

   So, if we say that $2 + \frac{1}{2} \approx 2$, then

   $$\frac{5}{12} \approx \frac{1}{2+\frac{1}{2}} = \frac{1}{\frac{5}{2}} = \frac{2}{5} .$$

   So we may actually see 5 spiral parastichies instead of the 12 straight parastichies if we don't have enough dots.

   (b) When the divergence angles are changed by 5^o describe the changes in the parastichies pattern you observe. Is the shape of the parastichies the same? In which cases do the number of parastichies stay the same and in which cases do the number of parastichies change? Explain mathematically.
   For 50^o there are 7 spiral parastichies, which spiral clockwise. This is because $7 \times 50 = 350$ which is slightly less than one complete revolution. Equivalently

   $$\frac{50}{360} = \frac{5}{36} = \frac{1}{\frac{36}{5}} = \frac{1}{7+\frac{1}{5}} \approx \frac{1}{7} .$$

   So there should be 7 spiral parastichies. For 95^o there are 4 parastichies that spiral clockwise because $4 \times 95 = 380$ which is slightly more than one complete revolution and for 125^o there are 3 parastichies that spiral clockwise because $3 \times 125 = 375$ which is slightly more than one complete revolution. For the next two it is helpful to work with continued fractions. For 140^o we note that

   $$\frac{140}{360} = \frac{7}{18} = \frac{1}{\frac{18}{7}} = \frac{1}{2+\frac{4}{7}} = \frac{1}{2+\frac{1}{\frac{7}{4}}} = \frac{1}{2+\frac{1}{1+\frac{3}{4}}} = \frac{1}{2+\frac{1}{1+\frac{1}{\frac{4}{3}}}} = \frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{3}}}}$$

So we have the approximate result

$$\frac{7}{18} \approx \frac{1}{2+\frac{1}{1+\frac{1}{1}}} = \frac{2}{5} .$$

So we expect 5 spiral parastichies. Finally, for 155^o

$$\frac{155}{360} = \frac{31}{72} = \frac{1}{\frac{72}{31}} = \frac{1}{2+\frac{10}{31}} = \frac{1}{2+\frac{1}{\frac{31}{10}}} = \frac{1}{2+\frac{1}{3+\frac{1}{10}}} \approx \frac{1}{2+\frac{1}{3}} = \frac{1}{\frac{7}{3}} = \frac{3}{7}$$

So there are 7 spiral parastichies.

(c) In the case where the divergence angle is 137.5^o you should be able to see two different ways to connect the dots to form parastichies. How many spirals do you get by each method of connecting the dots? Are these numbers familiar?
In my graph I get 8 counterclockwise spirals and 13 clockwise spirals. These are neighboring Fibonacci numbers.

(d) Of all the patterns which results in the arrangement that is most uniformly packed?
The divergence angle of 137.5^o gives the most efficient packing.

2. For the divergences angle of 137.5^o the number of spirals in each way of forming the parastichies should be neighbouring Fibonacci numbers. There is a mathematical reason for this

(a) For any given divergence angle α there is a complementary angle $360^o - \alpha$ which would generate an identical pattern. For example a divergence angle of 270^o generates the same pattern as a divergence angle of 90^o. Why?
A rotation of 270^o counter-clockwise is equivalent to a rotation of 90^o clock-wise.

(b) For the divergence of angle of 137.5^o find the complementary angle. Find the ratio of the complementary angle to the divergence angle in this case. What is this number? What is its connection to Fibonacci numbers?
The complementary angle is $\alpha = 360 - 137.5 = 222.5^o$. The ratio is $\dfrac{222.5}{137.5} = 1.618$ which is the Golden ration ϕ. Note: $\phi = \frac{F_{n+1}}{F_n}$, where F_n is the nth Fibonacci number.

3. Create a drawing on a blank piece of paper of the pattern that emerges for the divergence angle 137.5^o. Replace the dots in your diagram with some representation of a plant structure of your choice (eg seeds, scale, petals, or florets). Consider altering the size, shape and color of the structures as you move radially outward to make your pattern more natural. Be creative.

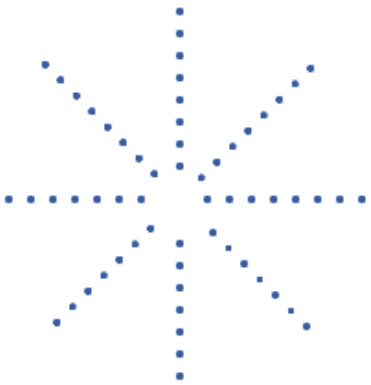
Spiral phyllotaxis is a characteristic arrangement of leaves, petals, scales or seeds that is seen in a variety of plants including daisies, sunflowers, pine cones and cauliflower. The cause of this beautiful natural pattern has only recently been explained fully. The explanation involves an interesting mix of mathematics, biology and physics. At the growth tip of a plant (called the **meristem**) small protusions called **primordia** emerge at regular intervals and move away radially from the center. Eventually these primordia go on to develop into various features of the plant such as its petals or seeds. In this workshop you will investigate how the primordia arrange themselves into a characteristic spiral pattern and why the number of spirals is so often a Fibonacci number.

(a) One model of growth for the primordia is to assume that each primordium emerges from the meristem at a fixed angle relative to the previous primordium. The angle separating neighbouring primordia is called the **divergence angle**. In this activity we will try out various divergence angles to see which gives the most efficient packing. We will do this by plotting points (or preferably coloured dots) representing primordia on polar graph paper. Plot the first dot at $\theta = 0$ and $r = 10$. For each subsequent point increase θ by the chosen divergence angle and increase r by one unit. **Do not connect the dots.** The spiral that is formed by connecting the dots together in the order they are plotted (i.e. in the order that the primordia grow) is called the **generating spiral** However, we are not interested in the generating spiral, but rather in the spirals or lines that emerge as our eyes make connections between nearest neighbour dots. These spirals are called **parastichies**.

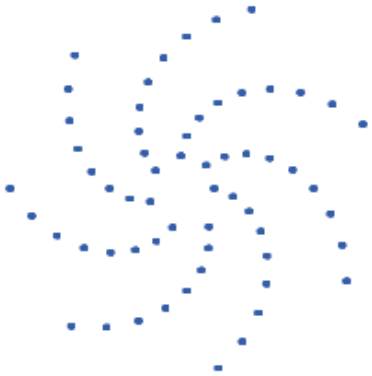
(a) Each member of the group should choose a different one of the following divergence angles, 45^o, 90^o, 120^o, 135^o and 150^o. Plot enough points so that the parastichies become evident (you will need as many as 36 in some cases) and in pencil connect the dots along each parastichy.

(b) Now each member of the group should repeat the exercise above but choose an angle that is 5^o larger than the one they chose at first.

(c) Finally everyone should repeat the exercise using a divergence angle of 137.5^o (it may be helpful to measure out this angle once on a triangular wedge to use as a template). You will likely need 36 or more primordia to see the parastichies.

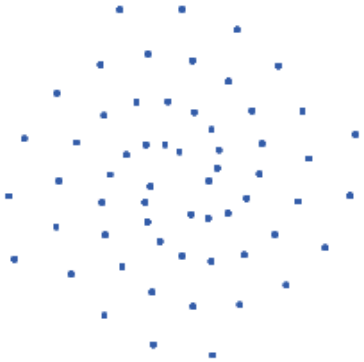


45o

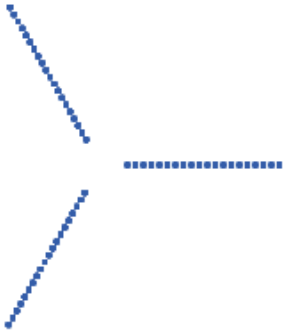


50o

90°

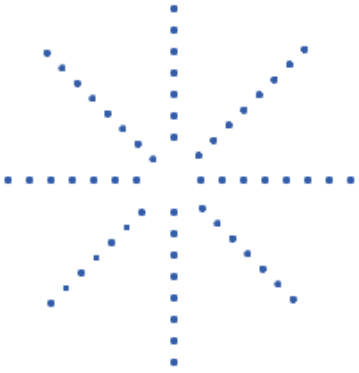


95°

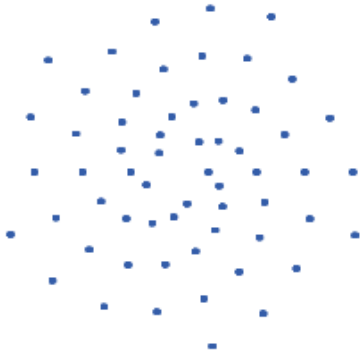


120°

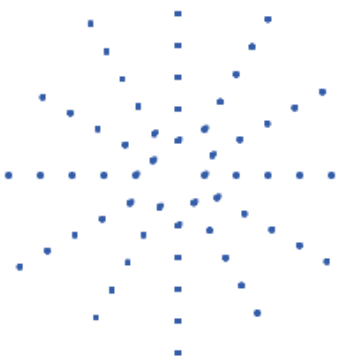
125°



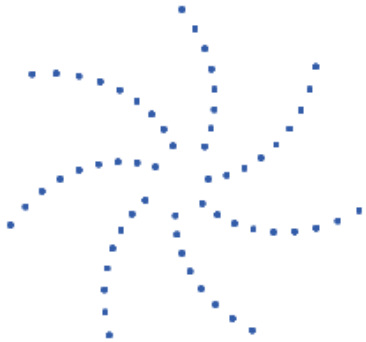
135°



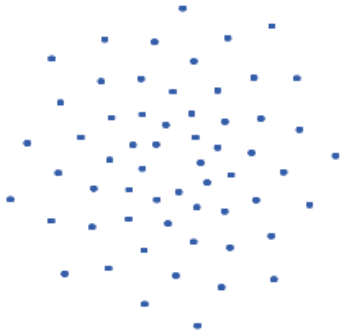
140°



150°



155°



137.5°

(b) Draw a rough sketch of each pattern of parastichies created by the members of the group. Record the divergence angle for each diagram carefully, and make sure you accurately represent the number and shape of the parastichies as you will use these results in your homework assignment.