A tessellation or tiling is a covering of the plane with non-overlapping figures. Given a particular figure or set of figures it is an interesting mathematical question to ask if they can be used to tile the plane, and if so how. In this worksheet we will investigate the number of ways we can tile the plane using regular polygons – ie those polygons with equal sides and angles. In particular we will look at regular and semi-regular tilings. In a regular tiling only one type of regular polygon is used. In a semi-regular tiling more than one type of regular polygon is used. In both cases the polygons must meet edge-edge and the configuration of polygons that meet at each vertex must be the same. We will examine what kind of configurations of regular polygons can fit around a vertex first. Since the sum of the interior angles of the polygons that meet at a vertex must sum to 360^o there is a limit to the number of configurations that are possible.

1. First it will be useful to determine the interior angles of suitable polygons. Fill out the table below to identify the interior angles of the possible polygons that could be used to tile the plane

Polygon	Sides	Angle
Triangle	3	60^o
Square	4	90^o
Pentagon	5	108^o
Hexagon	6	120^o
Heptagon	7	128.57^o
Octagon	8	135^o
Nonagon	9	140^o
Decagon	10	144^o
Dodecagon	12	150^o
Pentakaidecagon	15	156^o
Octakaidecagon	18	160^o
Icosagon	20	162^o
Tetrakaicosagon	24	165^o

2. Now we will start examining what vertex configurations are possible. At a vertex we must have at least three polygons. Why?
The interior angles of regular polygons are always less than 180^o. Therefore more than two of them are needed to have a total of 360^o at a vertex.

3. What is the largest number of regular polygons that can fit around a vertex? What polygon should be used?
The regular polygon with the smallest interior angle is the equilateral triangle with an angle of 60^o. Six times 60^o is 360^o. Therefore, a maximum of six of these can meet at a vertex.

4. You now have a rule which limits the number of polygons that can fit around a vertex. For another useful rule answer the following question. What is the maximum number of different types of regular polygon that can fit around a vertex?
Considering one each of the first four regular polygons (equilateral triangle, square, pentagon and hexagon) meeting at a common vertex. The interior angles adds up to $60+90+108+120 = 378$, which is more than 360^o. Thus, we can have at most three different types of polygons meeting at a point.

5. You are now in a position to systematically identify all the possible vertex configurations. In your groups fill out the following table using the table from question 1 and the hints below as a guide. You may find it helpful to cut out regular polygons to confirm your answers, although you are not required to do this.

Vertex Symbol	Number of Polygons		Vertex Symbol	Number of Polygons
3.3.3.3.3.3	6		3.7.42	3
3.3.3.3.6	5		3.8.24	3
3.3.3.4.4	5		3.9.18	3
4.4.4.4	4		3.10.15	3
3.3.6.6	4		3.12.12	3
3.3.4.12	4		4.5.20	3
3.4.4.6	4		4.6.12	3
6.6.6	3		5.5.10	3
4.8.8	3			

As is indicated in the table there is one way to fit six regular polygons at a vertex, two ways to fit five regular polygons around a vertex, four ways to fit four polygons around a vertex and 10 ways to fit 3 regular polygons around a vertex. Some of the values are given. Verify that these are correct. The following hints may help you identify those cases where there are three regular polygons around a vertex.

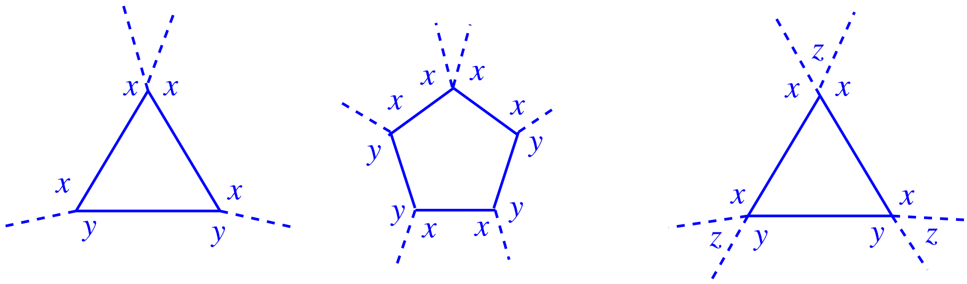
   (a) Besides the configurations 6.6.6 and 4.8.8 there are two others with duplicate polygons.
   (b) Two of the remaining configurations include a square.
   (c) The last three each contain one equilateral triangle.

6. Not all of the above vertex configurations are allowed in semi-regular tilings. This is because in a semi-regular tiling every vertex must have the same configuration. The following restrictions apply.

   (a) In a semi-regular tiling a vertex configuration $3.n.m$ is only possible if $n = m$.
   (b) In a semi-regular tiling a vertex configuration $5.n.m$ is only possible if $n = m$.
   (c) In a semi-regular tiling a vertex configuration $3.k.n.m$ is only possible if $k = m$.

Try to explain these rules by drawing diagrams and then use the rules to eliminate some of the possible vertex configurations above. Careful, in rule (c) the order of the polygons is important. Some arrangements of four polygons around a 3-vertex may be allowed and others may not.

The rules come about because the vertex configuration must be the same at all vertices. When a regular polygon has an odd number of sides this condition leads to a constraint as shown in the diagram below.



To read these diagrams, start at the top vertex and place polygons moving around counter clockwise until you reach the top again. To ensure that the same configuration occurs at each vertex we must have $x = y$ in each case.

Applying the rules means that of the vertex configurations with 3 polygons, only 6.6.6, 4.8.8, 3.12.12 and 4.6.12 will work. Of the ones with 4 polygons, only 4.4.4.4, 3.6.3.6, 3.4.6.4 are possible. The other configurations that are allowed are 6.6.6.6.6.6, 3.3.3.3.6, 3.3.3.4.4 and 3.3.4.3.4. Notice that the last two cases are two different vertex arrangements using the same polygons.