## 7.1 The Tension between Surface Area and Volume

Why do small animals have faster heartbeats and higher metabolic rates than large ones? Why are the shapes of the bodies and organs of large animals often quite different from those of small ones? To find answers to these questions, we examine how the relationship between surface area and volume changes as objects increase in size.

### Scaling Up a Cube

Let's look at what happens to the surface area and volume of a simple geometric figure, the cube, as we increase its size. In Figure 7.1 we have drawn a series of cubes where the lengths of the edges are 1, 2, 3, and 4 units.[1]

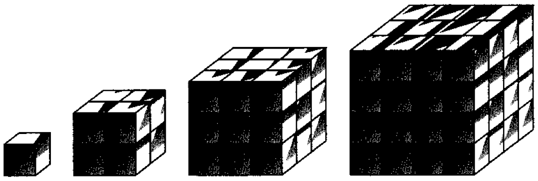


***Figure 7.1*** Four cubes for which the lengths of the edges are 1, 2, 3, and 4 units, respectively.

#### *Surface area of a cube*

If we were painting a cube, the surface area would tell us how much area we would have to cover. Each cube has six identical faces, so

$$\text{surface area} = 6 \cdot (\text{area of one face})$$

If the edge length of one face of the cube is $x$, then the surface area of that face is $x^2$. So the total surface area $S(x)$ of the cube is

$$S(x) = 6x^2$$

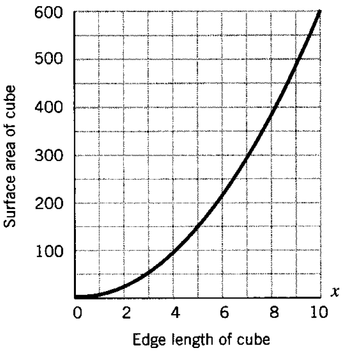


***Figure 7.2*** Graph of the surface area of a cube, $S(x) = 6x^2$.

The second column of Table 7.1 lists the surface areas of cubes for various edge lengths. Figure 7.2 shows a graph of the function $S(x)$.

Since the surface area of a cube with edge length $x$ can be represented as

$$\text{surface area} = \text{constant} \cdot x^2$$

we say that the surface area is *directly proportional* to the *square* (or second power) of the length of its edge. In Table 7.1, observe what happens to the surface area when we double the length of an edge. If we double the length from 1 to 2 units, the surface area becomes four times larger, increasing from 6 to 24 square units. If we double the length from 2 to 4 units, the surface area is again four times larger, increasing this time from 24 to 96 square units. In general, if we double the length of the edge from $x$ to $2x$, the surface area will increase by a factor of $2^2$, or 4:

$$\begin{aligned}
&\text{Surface area of a cube with edge length } x: & S(x) &= 6x^2\\
&\text{Surface area of a cube with edge length } 2x: & S(2x) &= 6(2x)^2\\
&\quad\text{If we apply rules of exponents} & &= 6 \cdot 2^2 \cdot x^2\\
&\quad\text{simplify and rearrange terms} & &= 4(6x^2)\\
&\quad\text{substitute } S(x) \text{ for } 6x^2\text{, we get} & S(2x) &= 4 \cdot S(x)
\end{aligned}$$

---

[1] For this discussion it doesn't matter which unit we use, but if you prefer, you may think of "unit" as being

**Ratio of Surface Area to Volume of Cube with Length of Edge $x$**

| Edge Length<br>$x$ | Surface Area<br>$S(x) = 6x^2$ | Volume<br>$V(x) = x^3$ | $\frac{\text{Surface Area}}{\text{Volume}}$<br>$\frac{S(x)}{V(x)} = \frac{6x^2}{x^3} = \frac{6}{x} = R(x)$ |
|---|---|---|---|
| 1 | 6 | 1 | 6.00 |
| 2 | 24 | 8 | 3.00 |
| 3 | 54 | 27 | 2.00 |
| 4 | 96 | 64 | 1.50 |
| 6 | 216 | 216 | 1.00 |
| 8 | 384 | 512 | 0.75 |
| 10 | 600 | 1000 | 0.60 |

**Table 7.1**

### Volume of a cube

If the edge length of one face of the cube is $x$, then the volume $V(x)$ of the cube is given by

$$V(x) = x^3$$

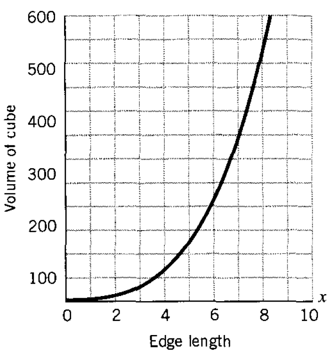


**Figure 7.3** Graph of the volume of a cube, $V(x) = x^3$.

The third column of Table 7.1 lists values for $V(x)$ and Figure 7.3 is the graph of $V(x)$. Since

$$\text{volume} = \text{constant} \cdot x^3 \qquad \text{(the constant in this case is 1)}$$

we say that the volume is *directly proportional* to the *cube* (or the third power) of the length of its edge.

What happens to the volume when we double the length? In Table 7.1, if we double the length from 1 to 2 units, the volume increases by a factor of $2^3$ or 8, increasing from 1 to 8 cubic units. If we double the length from 2 to 4 units, the volume again becomes eight times larger, increasing from 8 to 64 cubic units. In general, if we double the edge length from $x$ to $2x$, the volume will increase by a factor of 8:

| | |
|---|---|
| Volume of a cube with edge length $x$: | $V(x) = x^3$ |
| Volume of a cube with edge length $2x$: | $V(2x) = (2x)^3$ |
| If we apply rules of exponents | $= 2^3 \cdot x^3$ |
| simplify | $= 8x^3$ |
| substitute $V(x)$ for $x^3$, we get | $V(2x) = 8 \cdot V(x)$ |

### Surface area/volume

When we double the length of the edge, the surface area increases by a factor of 4, but the volume increases by a factor of 8. Starting with an edge length of 1, as we increase the edge length, the volume grows faster than the surface area. Hence the ratio

$$\frac{\text{surface area}}{\text{volume}}$$

decreases as the side length increases. See the fourth column of Table 7.1.

## Size and Shape

What we learned about the cube is true for any three-dimensional object, no matter what the shape. In general,

*For any shape, as an object becomes larger while keeping the same shape, the ratio of its surface area to its volume decreases.*

In Exploration 7.1 you can study further the effects of scaling up an object.

Thus a larger object has relatively less surface area than a smaller one. This fact allows us to understand some basic principles of biology and to answer the questions we asked at the beginning of this section.

Biological functions such as respiration and digestion depend upon surface area but must service the body's entire volume.[2] The biologist J. B. S. Haldane wrote that "comparative anatomy is largely the story of the struggle to increase surface in proportion to volume." This is why the shapes of the bodies and organs of large animals are often quite different from those of small ones. Many large species have adapted by developing complex organs with convoluted exteriors, thus greatly increasing the organs' surface areas. Human lungs, for instance, are heavily convoluted to increase the amount of surface area, thereby increasing the rate of exchange of gases. Stephen Jay Gould wrote that "the villi of our small intestine increase the surface area available for absorption of food (small animals neither have nor need them)."

Body temperature depends upon the ratio of surface area to volume. Animals generate the heat needed for their volume by metabolic activity and lose heat through their skin surface. Small animals have more surface area in proportion to their volume than do large animals. Since heat is exchanged through the skin, small animals lose heat proportionately faster than large animals and have to work harder to stay warm. Hence their heatbeats and metabolic rates are faster. As a result, smaller animals burn more energy per unit mass than larger animals.

**Stephen Jay Gould's essay "Size and Shape" in *Ever Since Darwin: Reflections in Natural History* offers an interesting perspective on the relationship between the size and shape of objects.**

## Algebra Aerobics 7.1

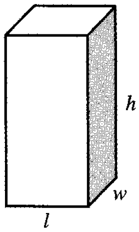
*For geometric formulas, see the back inside cover of this text.*

1. The function $C(r) = 2\pi r$ represents the circumference of a circle with radius $r$.
   - **a.** Find $C(R)$, $C(2R)$, and $C(3R)$. (Leave $\pi$ in answer)
   - **b.** By what factor is the circumference increased when *the radius* is doubled? When *the radius* is tripled?
2. The function $A(r) = \pi r^2$ represents the area of a circle with radius $r$.
   - **a.** Find $A(R)$, $A(2R)$, and $A(3R)$.
   - **b.** By what factor is the area increased when *the radius* is doubled? When *the radius* is tripled?
3. As the radius $r$ increases, what happens to the ratio $\frac{C(r)}{A(r)}$ from Problems 1 and 2?
4. The surface area and volume of a sphere are both functions of the radius. They can be described by the functions $v(r) = \frac{4}{3}\pi r^3$, where $v(r)$ represents the volume of a sphere with radius $r$, and $s(r) = 4\pi r^2$, where $s(r)$ represents the surface area of a sphere with radius $r$.
   - **a.** When you double the radius of a sphere, what happens to its surface area? To its volume?
   - **b.** Which eventually grows faster, the surface area or the volume? As a result, what happens to the surface area/volume ratio as the radius increases?
5. Compare a sphere with radius $r$ and a cube with edge length $r$.
   - **a.** For what value(s) of $r$, if any, is the volume of the cube equal to the volume of the sphere?
   - **b.** For what value(s) of $r$, if any, is the volume of the cube greater than the volume of the sphere? Less than?
6. Two different cylinders have the same height of 25 ft. but the base of one has a radius of 5 feet and the base of the other has a radius of 10 feet. Is the volume of one cylinder double the volume of the other? Justify your answer.

---

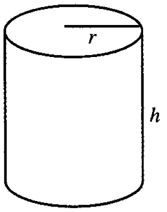
[2] For those who want to investigate how species have adapted and evolved over time, see D.W. Thompson. Growth and Form (New York: Dover, 1992), and T. McMahon and J. Bonner, *On Size and Life* (New Y

7. A cylindrical silo has a radius of 12 feet. If a certain amount of feed fills the silo to a depth of 5 feet, will twice the amount of feed fill the silo to a depth of 10 feet? Justify your answer.
8. The volume $V$ of some regular figures can be defined as the area $B$ of the base times the height $h$, or $V = Bh$.
   - **a.** Use this definition to find the formulas for the volumes in Figures 7.4–7.6.
   - **b.** In each case, if the height is doubled, by what factor is the volume increased?
   - **c.** If all dimensions are doubled, by what factor is the volume increased?



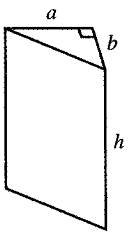
Rectangular prism

**Figure 7.4**



Cylinder

**Figure 7.5**



Triangular prism

**Figure 7.6**

## 7.2 *Direct Proportionality: Power Functions with Positive Powers*

In the last section we encountered three functions of the edge length $x$ of a cube:

| | |
|---|---|
| Surface area | $S(x) = 6x^2$ |
| Volume | $V(x) = x^3$ |
| Ratio of $\dfrac{\text{surface area}}{\text{volume}}$ | $R(x) = \dfrac{6}{x}$ or $6x^{-1}$ |

All three are called *power functions* since they are of the form

$$\text{dependent variable} = \text{constant} \cdot (\text{independent variable})^{\text{power}}$$

or

$$\text{output} = \text{constant} \cdot \text{input}^{\text{power}}$$

**Power Functions**

A power function $y = f(x)$ can be represented by an equation of the form

$$y = kx^p$$

where $k$ and $p$ are constants.

For the functions: $S(x) = 6x^2$, we have $k = 6$ and $p = 2$.
$V(x) = x^3$, we have $k = 1$ and $p = 3$.
$R(x) = 6x^{-1}$, we have $k = 6$ and $p = -1$.

**EXAMPLE 1** Decide whether or not each of the following functions is a power function of the form $y = kx^p$, and if it is, identify the values of $k$ and $p$.

**a.** $y = 3x^4$ **b.** $y = 3x^4 + 1$ **c.** $y = 2x^{-4}$ **d.** $y = -6\sqrt{x}$ **e.** $y = 4 \cdot 2^x$

**Solution**

**a.** $y = 3x^4$ is a power function where $k = 3$ and $p = 4$.

**b.** $y = 3x^4 + 1$ is not a power function since it cannot be written in the form $y = kx^p$.

**c.** $y = 2x^{-4}$ is a power function where $k = 2$ and $p = -4$.

**d.** $y = -6\sqrt{x} = -6x^{1/2}$ is a power function where $k = -6$ and $p = \frac{1}{2}$.

**e.** $y = 4 \cdot 2^x$ is not a power function, since the variable $x$ is in the exponent. It's an exponential function.

## Direct Proportionality

In Chapter 2, for linear functions, we said that $y$ is *directly proportional to* $x$ if $y$ equals a constant times $x$. For example, if $y = 4x$, then $y$ is directly proportional to $x$. We can extend the same concept to any power function with positive exponents. If $y = kx^p$ and $p$ is positive, we say that $y$ is *directly proportional to* $x^p$.

In Section 7.1, we saw that the surface area of a cube is directly proportional to the square of its edge length and that the volume is directly proportional to the cube of its edge length. The symbol $\propto$ is used to indicate direct proportionality. For example, if $y = 5x^3$, then $y$ is directly proportional to $x^3$, which we write as $y \propto x^3$.

**Direct Proportionality**

If

$$y = kx^p \qquad (k \neq 0 \text{ and } p > 0)$$

we say that $y$ is *directly proportional to* $x^p$. We write this as

$$y \propto x^p$$

The coefficient $k$ is called the *constant of proportionality.*

**EXAMPLE 2**

Write formulas to represent the following relationships.

**a.** The circumference, $C$, of a circle is directly proportional to its radius, $r$.

**b.** The area, $A$, of a circle is directly proportional to the radius, $r$, squared.

**c.** The volume, $V$, of a liquid flowing through a tube is directly proportional to the fourth power of the radius, $r$, of the tube.

**Solution**

**a.** $C = kr$, where $k = 2\pi$. **c.** $V = kr^4$ for some constant $k$.

**b.** $A = kr^2$, where $k = \pi$.

## Properties of Direct Proportionality

Consider a general power function $f(x) = kx^p$ where $p > 0$. If we double the inpu from $x$ to $2x$, we have

evaluating $f$ at $2x$ $\qquad f(2x) = k(2x)^p$

using rules of exponents $\qquad = k(2^p x^p)$

and rearranging terms $\qquad = 2^p(kx^p)$

substituting $f(x)$ for $kx^p$ $\qquad = 2^p f(x)$

**? SOMETHING TO THINK ABOUT**

*Weight is directly proportional to length cubed, but the ability to support the weight, as measured by the cross-sectional area of the bones, is proportional to length squared. Why does this mean that Godzilla or King Kong could exist only in the movies?*

Doubling the input multiplies the output by $2^p$. We saw in Section 7.1, if we double the input for the function $S(x) = 6x^2$ (the surface area of a cube), the output is multiplied by $2^2$ or 4.

If we multiply the input by $m$, changing the input from $x$ to $mx$, then

$$\begin{array}{ll} \text{evaluating } f \text{ at } mx & f(mx) = k(mx)^p \\ \text{using rules of exponents} & \phantom{f(mx)} = k(m^p x^p) \\ \text{and rearranging terms} & \phantom{f(mx)} = m^p(kx^p) \\ \text{substituting } f(x) \text{ for } kx^p & \phantom{f(mx)} = m^p f(x) \end{array}$$

So multiplying the input by $m$ multiplies the output by $m^p$. For example, if we triple the input for $S(x) = 6x^2$, the output is multiplied by $3^2$ or 9.

In general,

> If $y$ is directly proportional to $x^p$ (where $p > 0$), then $y = kx^p$ for some nonzero constant $k$.
>
> Multiplying the input by $m$ multiplies the output by $m^p$.
>
> For example, tripling the input multiplies the output by $3^p$.

### EXAMPLE 3

Given each function, what happens if the input is doubled? Increased by a factor of 10? Cut in half?

**a.** $y = 2x^4$ **b.** $h(z) = -2x^5$

**Solution**

**a.** If the input is doubled, the output is multiplied by $2^4$ or 16. If the input is multiplied by 10, the output is multiplied by $10^4$ or 10,000. If the input is cut in half, the output would be multiplied by $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

**b.** If the input is doubled, the output is multiplied by $2^5$ or 32. If the input is multiplied by 10, the output is multiplied by $10^5$ or 100,000. If the input is cut in half, the output would be multiplied by $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.

### EXAMPLE 4

**a.** What is the difference among $2f(x)$, $f(2x)$, and $f(x) + 2$?

**b.** If $f(x) = x^2$, evaluate the three expressions in part (a) when $x = 4$.

**Solution**

**a.** $2f(x)$ means to multiply the value of $f(x)$ by 2.
$f(2x)$ means to use $2x$ as the input for the function $f$.
$f(x) + 2$ means to evaluate $f(x)$ and then add 2.

**b.** If $f(x) = x^2$ and $x = 4$, then

$$\begin{aligned} 2f(4) &= 2 \cdot 4^2 = 2 \cdot 16 = 32 \\ f(2 \cdot 4) &= f(8) = 8^2 = 64, \text{ or equivalently,} \\ &= (2 \cdot 4)^2 = 2^2 \cdot 4^2 = 4 \cdot 16 = 64 \\ f(4) + 2 &= 4^2 + 2 = 16 + 2 = 18 \end{aligned}$$

**EXAMPLE 5** **Pedaling into the wind**

When you pedal a bicycle, it's much easier to pedal with the wind than into the wind. That's because as the wind blows against an object, it exerts a force upon it. This force is directly proportional to the wind velocity squared.

**a.** Construct an equation to describe the relationship between the wind force and the wind velocity.

**b.** If the wind velocity doubles, by how much does the wind force go up?

**Solution** **a.** $F(v) = kv^2$ for some constant $k$, where $v$ is the wind velocity and $F(v)$ is the wind force.

**b.** If the velocity doubles, then the wind force is multiplied by $2^2$ or 4. So pedaling into a 20-mph wind requires four times as much effort as pedaling into a 10-mph wind. See Figure 7.7.

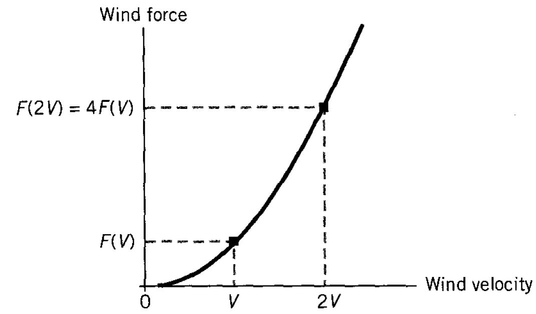


***Figure 7.7*** Doubling the wind velocity quadruples the wind force.

**EXAMPLE 6** **Earth's core**

The radius of the core of Earth is slightly over half the radius of Earth as a whole, ye the core is only about 16% of the total volume of Earth. How is this possible?

**Solution** We can think of both Earth and its core as approximately spherical in shape (see Figure 7.8).

If we let $r$ = radius of Earth, then Earth's volume is

$$V_{\text{Earth}} = \tfrac{4}{3}\pi r^3$$

If the radius of the core were exactly half that of Earth $\left(\frac{r}{2}\right)$, then the core's volume would be

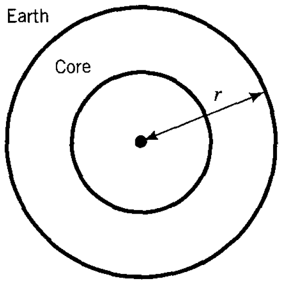


***Figure 7.8*** A sketch of a spherical Earth with radius $r$ and its core.

$$V_{\text{core}} = \tfrac{4}{3}\pi\left(\tfrac{r}{2}\right)^3$$

use rules of exponents $= \frac{4}{3}\pi\left(\frac{r^3}{2^3}\right)$

multiply and regroup terms $= \left(\frac{1}{2^3}\right)\left(\frac{4}{3}\pi r^3\right)$

substitute $\frac{1}{8}$ for $\left(\frac{1}{2^3}\right)$ and $V_{\text{Earth}}$ for $\frac{4}{3}\pi r^3$ $= \frac{1}{8}V_{\text{Earth}}$

$= 12.5\%$ of the volume of Earth

Since the radius of the core is slightly more than half the radius of Earth, 16% o Earth's volume is a reasonable estimate for the volume of the core.

**EXAMPLE 7** In Chapter 4 we encountered the following formula used by police. It estimates the speed, $S$, at which a car must have been traveling given the distance, $d$, the car skidded on a dry tar road after the brakes were applied:

$$S = \sqrt{30d} \approx 5.48d^{1/2}$$

Speed, $S$, is in miles per hour and distance, $d$, is in feet.

**a.** Use the language of proportionality to describe the relationship between $S$ and $d$.

**b.** If the skid marks were 50 feet long, approximately how fast was the driver going when the brakes were applied?

**c.** What happens to $S$ if $d$ doubles? Quadruples?

**d.** Is $d$ directly proportional to $S$?

**Solution** **a.** The speed, $S$, is directly proportional to $d^{1/2}$ and the constant of proportionality is 5.48.

**b.** If the length of the skid marks, $d$, is 50 feet, then the estimated speed $S \approx 5.48 \cdot (50)^{1/2} \approx 5.48(7.07) \approx 39$ mph.

**c.** If the skid marks double in length from $d$ to $2d$, then the estimate for the speed of the car increases from $\sqrt{30d}$ to $\sqrt{30 \cdot (2d)} = \sqrt{2} \cdot \sqrt{30d}$. So the estimated speed goes up by a factor of $\sqrt{2} \approx 1.414$ or, equivalently, by about 41.4%.

If the skid marks quadruple in length from $d$ to $4d$, then the estimated speed of the car goes from $\sqrt{30d}$ to $\sqrt{30 \cdot (4d)} = \sqrt{4} \cdot \sqrt{30d} = 2\sqrt{30d}$. So the speed estimate goes up by a factor of 2, or by 100%.

So if the skid marks doubled from 50 to 100 feet, the speed estimate would go up from 39 mph to $1.414 \cdot 39 \approx 55$ mph. If the skid marks quadrupled from 50 to 200 feet, then the speed estimate would double from about 39 to almost 78 mph.

**d.** To determine if $d$ is directly proportional to $S$, we need to solve our original equation for $d$.

| | |
|---|---|
| Given | $S = \sqrt{30d}$ |
| square both sides of the equation | $S^2 = 30d$ |
| divide both sides by 30 | $\dfrac{S^2}{30} = d$ |
| or | $d = \dfrac{S^2}{30}$ |

So $d$ is directly proportional to $S^2$ but not to $S$.

## Direct Proportionality with More Than One Variable

When a quantity depends directly on more than one other quantity, we no longer have a simple power function. For example, the volume, $V$, of a cylindrical can depends on both the radius, $r$, of the base and the height, $h$. The equation describing this relationship is

$$V = \text{area of base} \cdot \text{height}$$
$$V = \pi r^2 h$$

We say $V$ is directly proportional to both $r^2$ and $h$.

## 7.5 *Inverse Proportionality: Power Functions with Negative Integer Powers*

Recall that the general form of an equation for a power function is

$$\text{output} = \text{constant} \cdot (\text{input})^{\text{power}}$$

In Sections 7.1 through 7.4 we focused on functions where the power was a positive i teger. We now consider power functions where the power is a negative integer.

Using the rules for negative exponents, we can rewrite power functions in the fo

$$y = kx^{\text{negative power}}$$

where $k$ is a constant, as

$$y = \frac{k}{x^{\text{positive power}}}$$

For example, $y = 3x^{-2}$ can be rewritten $y = \frac{3}{x^2}$. In this form (with a positive power) i is easier to make calculations and to see what happens to $y$ as $x$ increases or decreas in value.

In Section 7.1 we constructed a function $R(x) = \frac{6}{x} = 6x^{-1}$ from the ratio of surface area to volume of a cube with edge length $x$. $R(x) = 6x^{-1}$ is a power functio where the power $(-1)$ is a negative integer. Table 7.5 and Figure 7.24 help us see th as $x$ increases, $R(x)$ decreases.

| Edge Length $x$ | $\frac{\text{Surface Area}}{\text{Volume}}$ $R(x) = \frac{6}{x}$ |
|---|---|
| 1 | 6.0 |
| 2 | 3.0 |
| 3 | 2.0 |
| 4 | 1.5 |
| 5 | 1.2 |
| 6 | 1.0 |
| 10 | 0.6 |

**Table 7.5**

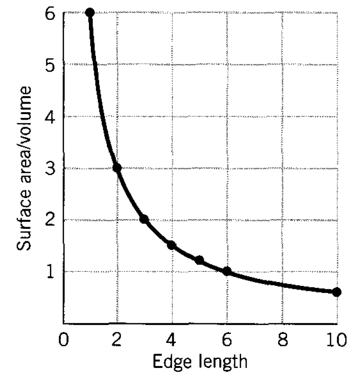


**Figure 7.24** The graph of $R(x) = \frac{6}{x}$ shows that the ratio of surface area/volume decreases as the edge length, $x$, increases.

### Inverse Proportionality

The headline for a March 8, 2004, article in *The Economic Times* read "Grace is inversely proportional to the crisis an individual faces." The author was arguing that the greater the crisis, the less gracious an individual is likely to be. Mathematics has a formal definition for the same concept.

For power functions in the form

$$y = \frac{\text{constant}}{x^p}$$

where $p$ is positive, we say that $y$ is *inversely proportional* to $x^p$. For example, if $y = \frac{8}{x^3}$, we say that $y$ is *inversely proportional to* $x^3$. If $y$ is inversely proportional to $x^p$, then as $x$ increases, $y$ decreases.

**Direct and Inverse Proportionality**

Let $p$ be a positive number and $k \neq 0$.

If $y = kx^p$, then $y$ is *directly proportional to* $x^p$.

If $y = \dfrac{k}{x^p} = kx^{-p}$, then $y$ is *inversely proportional to* $x^p$.

In both cases $k$ is called the *constant of proportionality*.

**EXAMPLE 1** **Examples of direct and inverse proportinality**

Write formulas to represent the following relationships.

**a.** Boyle's Law says that the volume, $V$, of a fixed quantity of gas is inversely proportional to the pressure, $P$, applied to it.

**b.** The force, $F$, keeping an electron in orbit is inversely proportional to the square of the distance, $d$, between the electron and the nucleus.

**c.** The acceleration, $a$, of an object is directly proportional to the force, $F$, applied upon the object and inversely proportional to the object's mass, $m$.

**Solution** **a.** $V = \frac{k}{P}$ for some constant $k$ **c.** $a = \frac{kF}{m}$ for some constant $k$

**b.** $F = \frac{k}{d^2}$ for some constant $k$

**EXAMPLE 2** In each formula, identify which variables are directly or inversely proportional to each other, and specify the constant of proportionality.

**a.** $a = \frac{v}{t}$, where $a$ = acceleration, $v$ = velocity, and $t$ = time.

**b.** $F = \dfrac{GM_1M_2}{d^2}$, Newton's Law of Universal Gravitation, where $F$ is the force of gravity, $G$ is a gravitational constant, $M_1$ and $M_2$ are the masses of two bodies, and $d$ is the distance between the two bodies.

**Solution** **a.** Acceleration, $a$, is directly proportional to velocity, $v$, and inversely proportional to time, $t$. The constant of proportionality is 1.

**b.** The force of gravity, $F$, is directly proportional to the masses of the two bodies, $M_1$ and $M_2$, and inversely proportional to $d^2$, the square of the distance between them. The constant of proportionality is $G$, a gravitational constant.

## Properties of Inverse Proportionality

If $y$ is *inversely proportional* to $x^p$, then $y = \frac{k}{x^p}$ (where $p > 0$) for some nonzero constant $k$. For an inversely proportional relationship, when we multiply the input by $m$, the output is multiplied by $\frac{1}{m^p}$ (where $p > 0$). Similar to our argument for direct proportionality, if $f(x) = \frac{k}{x^p}$, then

$$\begin{aligned}
&\text{evaluate } f \text{ at } mx && f(mx) = \frac{k}{(mx)^p} \\
&\text{use rules of exponents} && = \frac{k}{(m^p x^p)} \\
&\text{factor out } \frac{1}{m^p} && = \left(\frac{1}{m^p}\right) \cdot \left(\frac{k}{x^p}\right) \\
&\text{substitute } f(x) \text{ for } \left(\frac{k}{x^p}\right) && = \left(\frac{1}{m^p}\right) \cdot f(x)
\end{aligned}$$

Note that $k$, the constant of proportionality, does not play a role in these calculations.

For example, if $f(x) = \frac{3}{x^2}$, then doubling the input would multiply the output by $\frac{1}{2^2}$ or $\frac{1}{4}$. So if we double the input, the output would be reduced to one-fourth or 25% of the original amount. These doubling calculations do not depend on 3, the constant of proportionality.

If $y$ is inversely proportional to $x^p$, then $y = \frac{k}{x^p}$, where $p > 0$, and

multiplying the input by $m$ multiplies the output by $\frac{1}{m^p}$

For example, doubling the input multiplies the output by $\frac{1}{2^p}$.

## EXAMPLE 3

For each of the following functions, what happens if the input is doubled? Multiplied by 10? Cut in half?

**a.** $y = \frac{1}{x^4}$ **b.** $y = 2x^{-3}$

**Solution**

**a.** If the input is doubled, the output is multiplied by $\frac{1}{2^4} = \frac{1}{16}$ or 0.0625. If the input is multiplied by 10, the output is multiplied by $\frac{1}{10^4} = \frac{1}{10{,}000} = 0.0001$. If the input is cut in half, the output is multiplied by $\frac{1}{(0.5)^4} = \frac{1}{0.0625} = 16$.

**b.** The function $y = 2x^{-3}$ can be rewritten as $y = \frac{2}{x^3}$. In this form it's easier to see that if the input is doubled, the output is multiplied by $\frac{1}{2^3} = \frac{1}{8} = 0.125$. If the input is multiplied by 10, the output is multiplied by $\frac{1}{10^3} = \frac{1}{1000} = 0.001$. If the input is cut in half, the output is multiplied by $\frac{1}{(0.5)^3} = \frac{1}{0.125} = 8$.

## EXAMPLE 4

**The cardinal rule of scuba diving**

The most important rule in scuba diving is "Never, ever, hold your breath." Why?

**Solution**

Think of your lungs as balloons filled with air. As a balloon descends underwater, the surrounding water applies pressure, compressing the balloon. As the balloon ascends, the pressure is lessened and the balloon expands until it attains its original size when it reaches the surface. The volume $V$ of the balloon is inversely proportional to the pressure $P$; that is, as the pressure increases, the volume decreases. The relationship is given by a special case of Boyle's Law for Gases,

$$V = \frac{1}{P} \qquad \text{or equivalently} \qquad V = P^{-1}$$

where $V$ is the volume in cubic feet and $P$ the pressure measured in atmospheres (atm). (One atm is 15 $\text{lb/in}^2$, the atmospheric pressure at Earth's surface.) Each additional 33 feet of water depth increases the pressure by 1 atm. Table 7.6 and Figure 7.25 describe the relationship between pressure and volume.

**Pressure vs. Volume**

| Depth (ft) | Pressure (atm) | Volume ($\text{ft}^3$) |
|---|---|---|
| 0 | 1 | 1 |
| 33 | 2 | 1/2 |
| 66 | 3 | 1/3 |
| 99 | 4 | 1/4 |

***Table 7.6***

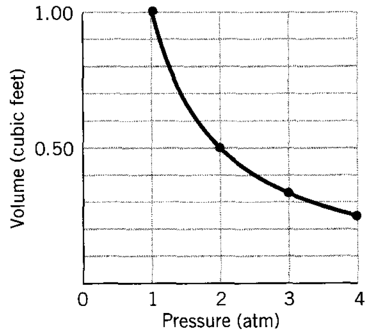


***Figure 7.25*** Graph of volume vs. pressure for a balloon descending underwater.

We can see in Table 7.6 that if we start with 1 cubic foot of air in the balloon at the surface and descend 33 feet, the surrounding pressure has doubled, from 1 to 2 atm, and the volume of air is cut in half. If the balloon descends to 99 feet, doubling the pressure again from 2 to 4 atm, the volume of air is cut in half again—leaving only one-fourth of the original volume. Why does this matter to divers?

Suppose you are swimming in a pool, take a lung full of air at the surface, and then dive down to the bottom. As you descend, the buildup of pressure will decrease the volume of air in your lungs. When you ascend back to the surface, the volume of air in your lungs will expand back to its original size and everything is fine.

But when you are scuba diving, you are constantly breathing air that has been pressurized at the surrounding water pressure. If you are scuba diving 33 feet below the surface of the water, the surrounding water pressure is at 2 atm, twice that at the surface. What will happen then if you fill your lungs from your tank, hold your breath, and ascend to the surface? When you reach the surface, the pressure will drop in half, from 2 atm down to 1 atm, so the volume of air in your lungs will double, rupturing your lungs! Hence the first rule of scuba diving is "Never, ever, hold your breath."

### EXAMPLE 5

**Designing a soda can**

A designer is asked to redesign a 12-ounce soda can. The volume must remain constant at 22 cubic inches (just enough to hold the 12 ounces and a little air) and the shape must remain cylindrical. What are her options?

**Solution** If $r$ = radius of the can (in inches) and $h$ = height (in inches), then

$$\begin{aligned}\text{volume of can} &= \text{area of base} \cdot \text{height of can}\\ &= \pi r^2 h\end{aligned}$$

Since the volume of the can must be 22 cubic inches,

$$22 = \pi r^2 h$$

Solving for $h$ gives $h = \frac{22}{\pi r^2}$

So $h$ is inversely proportional to $r^2$. If we substitute an approximation of 3.14 fo $\pi$, we get

$$h \approx \frac{22}{3.14 r^2} \approx \frac{7}{r^2}$$

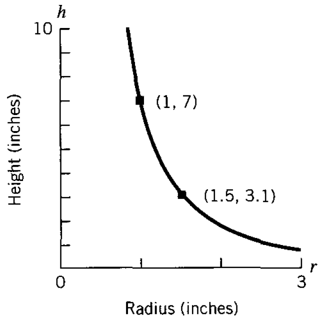


***Figure 7.26*** The relationship between height, $h$, and radius, $r$, of a can holding 22 cubic inches.

Figure 7.26 shows a graph of the relationship between the height of the can and its radiu

The designer can pick any point on the curve to determine the potential dime sions of the can. For example, if $r = 1$, then $h = 7$. So the point (1, 7) on the cur represents a radius of 1″ (hence a diameter of 2″) and a height of 7″. If $r = 1.5$, th $h = \frac{7}{(1.5)^2} = \frac{7}{2.25} \approx 3.1$ The point (1.5, 3.1) on the curve represents a radius of 1.5 (diameter of 3″) and a height of 3.1″. The two points are labeled on the graph (Figu 7.26), and the corresponding can sizes are drawn in Figure 7.27.

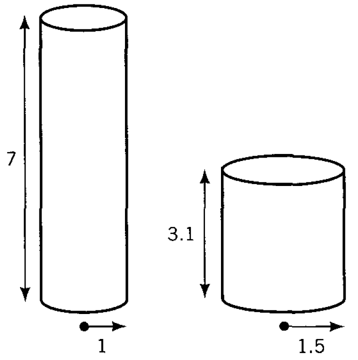


***Figure 7.27*** Two possible sizes for the soda can.

## Inverse Square Laws

When the output is inversely proportional to the square of the input, the functional relationship is called an *inverse square law*. In the preceding example, where we held the volume of the soda can fixed at 22, the resulting function $h = 7/r^2$ is an inverse square law. Inverse square laws are quite common in the sciences.

**EXAMPLE 6** **Seeing the light**

The intensity of light is inversely proportional to the square of the distance between the light source and the viewer.

**a.** Describe light intensity as a function of the distance from the light source.

**b.** What happens to the intensity of the light if the distance doubles? Triples?

**Solution** **a.** If $d$ is the distance from the light source, then $I(d)$, the intensity of the light, is inversely proportional to $d$, so we can write $I(d) = \frac{k}{d^2}$ for some constant $k$.

**b.** If the distance doubles, then the light intensity output is multiplied by $\frac{1}{2^2} = \frac{1}{4}$. So the intensity drops to one-fourth of the original intensity.

If the distance triples, then the light intensity is multiplied by $\frac{1}{3^2} = \frac{1}{9}$. So the intensity drops to one-ninth of the original intensity.

For example, if you are reading a book that is 3 feet away from a lamp, and you move the book to 6 feet away (doubling the distance between the book and the light), the light will be one-fourth as intense. If you move the book from 3 to 9 feet away (tripling the distance), the light will be only one-ninth as intense. The reverse is also true; for example, if the book is 6 feet away and the light seems too dim for reading, by cutting the distance in half (to 3 feet), the illumination will be four times as intense.

### EXAMPLE 7 Gravitational force between objects

The gravitational force between you and Earth is inversely proportional to the square of the distance between you and the center of Earth.

**a.** Express this relationship as a power function.

**b.** What happens to the gravitational force as the distance between you and the center of Earth increases by a factor of 10?

**Solution** **a.** The power function

$$F(d) = \frac{k}{d^2} = kd^{-2}$$

describes the gravitational force $F(d)$ between you and Earth in terms of a constant $k$ times $d^{-2}$, where $d$ is the distance between you and the center of Earth.

**b.** If the distance between you and Earth's center increases by factor of 10, then the gravitational force is multiplied by $\frac{1}{10^2} = \frac{1}{100}$; that is, multiplying the distance by 10 decreases the gravitational force to one-hundredth of its original size.

Suppose you were an astronaut who started at the surface of Earth (roughly 4000 miles from Earth's center) and traveled to 40,000 miles from Earth's center. The pull of Earth's gravity there would be one-hundredth that on Earth's surface. That's why astronauts appear weightless.

### EXAMPLE 8 Why many inverse square laws work

Inverse square laws in physics often depend on a power source and simple geometry. Imagine a single point as a source of power, emitting perhaps heat, sound, or light. We can think of the power radiating out from the point as passing through an infinite number of concentric spheres. The farther away you are from the point source, the lower the intensity of the power, since it is spread out over the surface area of a sphere that increases in size as you move away from the point source. Therefore, the intensity, $I$, of