1. (a) Given that $x = \log 3$, find the value of $y = 10^x$ without using a calculator.
   $y = 10^{\log 3} = 3$

   (b) Given that $x = 4\log 3$, find the value of $y = 10^x$ without using a calculator.
   $y = 10^{4\log 3} = 10^{\log 3^4} = 3^4 = 81$

2. Find the inverse of the following functions. That is find an expression for x as a function of y

   (a) $y = 10^{3x}$

   $$\Rightarrow\ 3x = \log y\ \Rightarrow x = \frac{\log y}{3}$$

   (b) $y = 2(5)^{3x+1}$

   $$\Rightarrow\ 5^{3x+1} = \frac{y}{2}\ \Rightarrow\ 3x + 1 = \log_5\left(\frac{y}{2}\right)\ \Rightarrow x = \tfrac{1}{3}\ \log_5\left(\frac{y}{2}\right) - \tfrac{1}{3}$$

   (c) $y = 2\log 5x - 1$

   $$\Rightarrow\ y + 1 = 2\log 5x\ \Rightarrow\ \log 5x = (y+1)/2\ \Rightarrow\ 5x = 10^{(y+1)/2}\ \Rightarrow\ x = \frac{1}{5}\,10^{(y+1)/2}$$

   (d) $y = 8x^3$

   $$\Rightarrow x^3 = \frac{y}{8} \Rightarrow x = (\frac{y}{8})^{\frac{1}{3}}\ \Rightarrow\ x = \frac{\sqrt[3]{y}}{2}$$

3. Expand $\log\left(\frac{xy}{z^3}\right)$ as far as possible using the properties of logs.

   $$\log\left(\frac{xy}{z^3}\right) = \log(xy) - \log(z^3)\ =\ \log x + \log y - 3\log z$$

4. The mouse population in a farm in Australia grew exponentially from 30 to 72 in 3 months. Write down an exponential function for this population and then find out how long will it take for the mouse population to reach 900.

   The growth factor for 3 months is $\frac{72}{30} = 2.4$ so if y is the population and x is months since the population was 30 then $y = 30(2.4)^{x/3}$. To find when the population is 900 we set $y = 900$ and solve for x. Hence

   $$900 = 30(2.4)^{x/3}\ \Rightarrow\ 30 = (2.4)^{x/3}\ \Rightarrow x/3 = \frac{\log(30)}{\log(2.4)} = 3.89\ \Rightarrow\ x = 11.6$$

   So it takes about 12 months.

5. Every action is risky to some extent. Most people would consider some activities, such as hang gliding or mountain climbing, quite risky. There are other things that are not commonly thought of as risky, such as driving to school or taking a shower, yet even these carry some risk. The probability of various risks has been calculated, but most of the numbers are so small that many people do not have an intuitive understanding of them. To help better understand these types of probabilities, John and Sean Paling in their book *Up to your Armpits in Alligators? How to Sort Out What Risks Are Worth Worrying About*, devised a logarithmic scale to describe risks. This scale uses common logarithms to convert a probability to a risk number.

Probability of an Event Occurring	Risk Number
1/10	5
1/100	4
1/1000	3
1/10,000	2
1/100,000	1
1/1,000,000	0
1/10,000,000	-1

Note that a one in a million chance is assigned a risk number of 0. Events associated with risk numbers below zero are insignificant and should not be of much concern. For example, the Food and Drug Administration is not concerned about the cancer risk from a particular food additive if the risk number is below zero. Even events that have a risk number of 0, +1, or +2 aren't very significant. For events with a risk number above two, however, you should be concerned about possible dangers. For instance, doctors now say that about one in nine women will get breast cancer. This is about a risk of +5.

(a) Find a function which expresses risk number r in terms of probability p. It may help to express each of the numbers in the first column of the table above as powers of 10.

$r = \log p + 6$

(b) The following are annual risks for Americans expressed as probabilities.

You will be injured if you play basketball regularly.	1/40
You will die of cancer.	1/500
You will be killed by a tornado.	1/2,000,000
You will contract the plague.	1/25,000,000

i. Find the risk number for each event. (up to one decimal place)

Basketball injury $r = 4.4$. Cancer $r = -3.3$. Tornado $r = -0.3$. Plague $r = -1.4$.

ii. Find the difference between the risk numbers for being injured playing basketball and dying of cancer. What does the difference in risks numbers tell you?
The difference is $4.4 - 3.3 = 1.1$ which means there is 1.1 order of magnitude greater risk (factor of $10^{1.1}$) of getting injured playing basketball then dying of cancer.

iii. Find the difference between the risk numbers for being killed by a tornado and contracting the plague. Is the difference between the first two events closer together, about the same, or further apart than the second two? Were you surprised at the comparison using risk numbers?
The difference is $-0.3 - (-1.4) = 1.1$. This difference is about the same, which may seem surprising.

(c) Use your expression for risk number as a function of probability from above to find an expression for probability as a function of risk number.
$p = 10^{r-6}$

(d) The following are annual risks for Americans given as risk numbers.

You will be injured and require immediate medical attention.	+5.5
You will have a fatal accident if you are a skydiver.	+3.0
An asteroid will collide with the earth.	+0.6
The extra risk of cancer from eating a char-broiled steak once a week.	-0.4.

the probability of each event above. Write your answers in the form: 1 in ________ chance.
Medical attention $p = 10^{-0.5} = 0.316 \Rightarrow$ a one in 3.16 chance. Fatal skydive $p = 10^{-3} = 0.001 \Rightarrow$ a one in 1000 chance. Asteroid $p = 10^{-5.4} = 3.98 \times 10^{-6} \Rightarrow$ a one in 250,000 chance. Cancer from steak $p = 10^{-6.4} = 3.98 \times 10^{-7} \Rightarrow$ a one in 2,500,000 chance.

(e) They say that lightning never strikes twice in the same place. However, it has been known to strike some people more than once in their lifetime.

i. The probability that lightning will strike you in your lifetime is 1 in 9000. What is the probability that it will strike you twice, assuming independence?
The probability is square so $\frac{1}{9000}\frac{1}{9000} = 1.23 \times 10^{-8}$.

ii. Find the risk numbers for being struck by lightning once in your life and twice in your life.
Probability of getting hit once: $r = 6 + \log(1/9000) = 2.05$ and $r = 6 + \log(1.23 \times 10^{-8}) = -1.9$.

iii. Challenge: Find a simple rule that will take a risk number associated with an event occurring once and transform it into the risk number of the same event occurring twice.
$r = 6 + \log p$ so if $p \to p^2$ then $r \to 6 + \log(p^2) = 6 + 2\log p = 2(6 + \log p) - 6 = 2r - 6$.

6. James, an ulcer patient, has been told to avoid acidic foods. If he drinks coffee, which has a pH of 5.0 it bothers him. His tap water with a pH of 5.8 is ok, as is milk, which has a pH of 6.9. Note pH is defined as pH$= -\log{[\mathrm{H}^+]}$ where $[\mathrm{H}^+]$ is the concentration of the H^+ ion.

   (a) Find the $[H^+]$ concentration in the coffee, water and milk.

   pH$= -\log{[\mathrm{H}^+]} \;\Rightarrow\; [\mathrm{H}^+] = 10^{-\mathrm{pH}}$.
   For coffee $[\mathrm{H}^+] = 10^{-5}$ M. For tap water $[\mathrm{H}^+] = 10^{-5.8} = 1.58 \times 10^{-6}$ M.
   For milk $[\mathrm{H}^+] = 10^{-6.9} = 1.26 \times 10^{-7}$ M

   (b) Find the $[H^+]$ concentration of a mixture which is half milk and half coffee. What is the pH of this mixture? Why is this not just the average of the two pH's?

   The $[\mathrm{H}^+]$ concentration of the mixture is the average concentration of the milk and coffee which is$(10^{-5} + 1.26 \times 10^{-7})/2 = 5.1 \times 10^{-6}$ M. So the pH is $-\log{(5.1 \times 10^{-6})} = 5.3$. This is not the average of the two pH's because a log scale is not linear. This is also more acidic than tap water.

   (c) If James wishes to make a 200 ml mixture of milk and coffee which has the same pH as tap water how much milk must he use? Will this coffee taste nice? (Hint: if x is the volume of milk that is used then the volume of coffee is $200 - x$)

   $C_{\mathrm{coffee}}V_{\mathrm{coffee}} + C_{\mathrm{milk}}V_{\mathrm{milk}} = C_{\mathrm{water}}V_{\mathrm{water}} \;\Rightarrow\; C_{\mathrm{coffee}}(200 - x) + C_{\mathrm{milk}}(x) = 200C_{\mathrm{water}}$
   $\Rightarrow\; 200C_{\mathrm{coffee}} - xC_{\mathrm{coffee}} + xC_{\mathrm{milk}} = 200C_{\mathrm{water}} \;\Rightarrow x(C_{\mathrm{milk}} - C_{\mathrm{coffee}}) = 200C_{\mathrm{water}} - 200C_{\mathrm{coffee}}$
   $\Rightarrow\; x = \dfrac{200(C_{\mathrm{water}} - C_{\mathrm{coffee}})}{C_{\mathrm{milk}} - C_{\mathrm{coffee}}} = \dfrac{200(1.58 \times 10^{-6} - 10^{-5})}{1.26 \times 10^{-7} - 10^{-5}} = 170$ ml of milk. The coffee will taste like milk!