1. Find the inverse of the function $y = 2e^{3x}$.
   First divide both sides by 2 to get $\frac{1}{2}y = e^{3x}$ and then take the natural log of both sides to get $\ln\left(\frac{1}{2}y\right) = 3x$ and finally $x = \frac{1}{3}\ln\left(\frac{1}{2}y\right)$. So the inverse function is $y = \frac{1}{3}\ln\left(\frac{1}{2}x\right)$.

2. Simplify the expression $y = 5e^{4\ln x}$.
   $4\ln x = \ln\left(x^4\right)$ so $y = 5e^{\ln(x^4)} = 5x^4$.

3. Suppose x and y both grow exponentially as $x = e^{\frac{1}{3}t}$ and $y = 2e^{0.4t}$.

   (a) From the expression for x as a function of t, find t as a function of x.
   Taking the natural log of both sides we get $\frac{1}{3}t = \ln x$ so that $t = 3\ln x$.

   (b) Substitute this value of t into the expression for y and hence find y as a function of x. Express this function as a power function, identifying the power and the proportionality constant.
   Substituting in for t we get $y = 2e^{(0.4)(3\ln x)} = 2e^{1.2\ln x} = 2e^{\ln\left(x^{1.2}\right)} = 2x^{1.2}$. So the power is 1.2 and the proportionality coefficient is 2.

4. The surface area of a spherical cell of radius r is $S = 4\pi r^2$ and its volume is $V = \frac{4}{3}\pi r^3$.

   (a) Find an expression for the surface area to volume ratio, $\frac{S}{V}$, and express your answer as a power function of r.
   $$\frac{S}{V} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{1}{\frac{1}{3}r} = \frac{3}{r} = 3r^{-1}.$$

   (b) If the radius of the cell doubles in size, what happens to the surface area to volume ratio?
   If $r \rightarrow 2r$ then $\frac{3}{r} \rightarrow \frac{3}{2r} = \frac{1}{2}\frac{3}{r} = \frac{1}{2}\frac{S}{V}$. So the surface to volume ratio is halved.

   (c) The amount of nutrients a cell needs is proportional to its volume, but the amount of nutrients a cell can collect is proportional to its surface area. Use this fact and your answer to the above questions to explain what happens to a cell's ability to sustain itself as it gets larger? Discuss how a cell can adapt in order to grow larger and still get the nutrients it needs?
   As the cell grows you would want the the nutrients collected to be greater than that required. In other words, you want the the ratio to always remain bigger than one. But this ratio is proportional to the surface to volume ratio which steadily decreases as the cell grows. Thus, eventually the cell would not be able to get enough resources to survive. The cell might adapt by changing its shape to increase the surface area to volume ratio by developing folds or becoming non-spherical.

5. The lift F provided by the wing of bird is directly proportional to the square of its wingspan L, but the mass M of the bird (which the wings need to support) is directly proportional to the cube of the wingspan. This difference leads to a limit on the size of birds which can fly.

   (a) Write the two proportionalities as power functions of the length of the wingspan.
   The first is $F = kL^2$ and the second is $M = cL^3$, where the two different proportionality constants are k and c.

   (b) Determine the proportionality coefficients, with units, for a 13.5 g chickadee which can generate enough lift to support 22.5 g if the wingspan is 15 cm.
   $k = \dfrac{F}{L^2} = \dfrac{22.5 \text{ g}}{(15 \text{ cm})^2} = \dfrac{22.5}{225} = 0.1 \text{ g/cm}^2.$
   Similarly $c = \dfrac{M}{L^3} = \dfrac{13.5 \text{ g}}{(15 \text{ cm})^3} = \dfrac{22.5}{3375} = 0.004 \text{ g/cm}^3.$

   (c) If a mad geneticist breeds a giant chickadee with wingspan 50 cm what will the mass and lift be, assuming the giant chickadee has the same proportions as the smaller one? Can the bird fly?
   The mass would be $M =$(0.004 g/cm^3)(50 cm)^3 = 500 g. The lift would be $L =$(0.1 g/cm^2)(50 cm)^2 = 250 g. The lift would only be able to support a 250 g "chickadee" so the bird would not fly.

   (d) What is the wingspan of the largest chickadee that could fly?
   The largest chickadee that could fly would be the one for which the lift was exactly equal to the mass. Setting the two functions equal we get $kL^2 = cL^3$. Solving for L, we get $L = \dfrac{k}{c} = \dfrac{0.1 \text{ g/cm}^2}{0.004 \text{ g/cm}^3} = 25$ cm. Actually the bird would have to be smaller than this since the bird would need more lift than this to get off the ground and sustain the flight.

6. For each of the following expressions of proportionality write down a single equation that relates the variables. Describe what happens to the first variable when the second variable is doubled.

   (a) The skidding distance d is directly proportional to the square of the speed of a car v.
   $d = kv^2$ so if speed is doubled $v \to 2v$ then $d \to 2^2 d = 4d$. Thus distanced is increased by a factor of 4.

   (b) The intensity of light, I, produced by a hot body is proportional to the fourth power of the temperature, T, in Kelvin.
   $I = kT^4$ so if $T \to 2T$ then $I \to 2^4 I$. Thus the intensity increases by a factor of 16.

   (c) The frequency of vibration f of a ball on the end of a spring is inversely proportional to the square root of the mass of the ball m.
   $f = \dfrac{k}{\sqrt{m}} = km^{-\frac{1}{2}}$ so if mass is doubled $m \to 2m$ then $f \to 2^{-\frac{1}{2}} f = .707f$. Thus frequency is decreased to about 71% of its original value.

7. As a person grows they get taller, wider and more voluminous. Although these measure are related, they do not change at the same rate or in the same proportion. Consider a cylindrical person. The formula for the volume of a cylinder is $V = \pi r^2 h$, where r is the radius of the base and h is the height.

   (a) Express the dependence of volume on radius and height as two proportionality relationships.
   Volume is directly proportional to the height and to the square of the radius.

   (b) What happens to the volume of the cylinder if the radius is doubled but the height remains the same?
   Since volume is proportional to radius squared then the volume is quadrupled when radius is doubled. ie $r \to 2r \Rightarrow V \to 2^2 V = 4V$.

   (c) What happens to the volume of the cylinder the height is doubled but the radius remains the same?
   Since volume is proportional to height then volume is doubled when height is doubled. ie $h \to 2h \Rightarrow V \to 2V$.

   (d) What happens to the volume of the cylinder both the height and the radius are doubled?
   $h \to 2h$ and $r \to 2r$ then $V \to 2 \times 2^2 V = 8V$. So volume is increased by a factor of 8.

   (e) Suppose that when the height has doubled the volume increases by a factor of 4. By what factor has the radius increased? (This last question relates to the growth of human children whose height grows faster than width in such a way that mass is proportional to height squared.)
   Suppose radius is increased by a factor of f then $h \to 2h$ and $r \to fr \Rightarrow V \to 2f^2 V = 4V$ so $2f^2 = 4 \;\Rightarrow\; f^2 = 2 \Rightarrow f = \sqrt{2} = 1.41$. So the radius must be increased by 41%.

8. The rate y at which blood flows through a blood vessel of radius r satisfies the power relationship

$$y = cr^4$$

   (a) If y is measured in ml per second and radius r is measured in cm, find the units for c.
   $c = y/r^4$. Therefore, the dimensions of c are $\mathrm{ml}/(\mathrm{cm})^4$.

   (b) If a main blood vessel branches into smaller blood vessels which are half the radius, how many branches do you need so that the total flow rate in the branches remains the same as that in the main blood vessel?
   If the radius is half then $r \to \frac{1}{2} r$ and hence $y \to \frac{1}{2}^4 y$. The flow rate is reduced to one sixteenth the original amount. You would there fore need sixteen branches of that size to keep the total flow rate the same.