1. For each of the following difference equations, calculate P_1, P_2 and P_3. Then find the the general formula for P_t and hence find when the population is expected to reach 1000.

   (a) $P_{t+1} = P_t + 5$, $P_0 = 20$.
   $P_1 = P_0 + 5 = 25$, $P_2 = 30$, and $P_3 = 45$. In general $P_t = 20 + 5t$. So $P_t = 1000$ when $1000 = 20 + 5t \Rightarrow 980 = 5t \Rightarrow t = 196$.

   (b) $P_{t+1} = 1.5P_t$, $P_0 = 8$.
   $P_1 = (1.5)P_0 = (1.5)(8) = 12$, $P_2 = (1.5)(12) = 18$, and $P_3 = (1.5)(18) = 27$.
   The general formula is $P_t = 8(1.5)^t$. So $P_t = 1000$ when $1000 = 8(1.5)^t \Rightarrow 125 = (1.5)^t \Rightarrow t = \ln(125)/\ln(1.5) \approx 12$.

2. For the difference equation $M_{t+1} = 0.5M_t + 1$,

   (a) find M_t at t =1,2,3,4 and 5 if the initial value is $M_0 = 10$.
   $M_1 = 0.5M_0 + 1 = 0.5(10) + 1 = 6$. $M_2 = 0.5M_1 + 1 = 0.5(6) + 1 = 4$. Similarly $M_3 = 3$, $M_4 = 2.5$, and $M_5 = 2.25$.

   (b) What do you think the the value of M_t will be after a long time?
   It looks like M_t reaches and equilibrium of 2.

   (c) Challenge: Using any pattern you see in your answers to (a) and your answer to (b) above, find an expression for M_t as a function of time t.
   It looks like an exponential decay down to the value 2.
   $M_0 = 8 + 2$, $M_1 = 4 + 2$, $M_2 = 2 + 2$, $M_3 = 1 + 2$, $M_4 = 0.5 + 2$, and $M_4 = 0.25 + 2$.
   The pattern is $M_t = 8(0.5)^t + 2$.

3. A female mouse reproduces approximately once every two months for the duration of her life which is approximately one year. She produces a litter size of about 8 (4 males and 4 females) on average, under ideal conditions.

   (a) Estimate the birth rate, death rate and per capita growth rate of mice choosing months as your time unit. Discuss your assumptions and answers with your group.
   In her life time a female produces about 4 females every two months for a year, which is 24 females a year, so the birth rate is 24/12=2 per month. The death rate is approximately the reciprocal of the life span which is (1/12)=0.0833 per month. Thus the per capita growth rate is $r = b - d = 2 - 0.833 \approx 1.9$.

   (b) Assuming the per capita growth rate from above estimate how long it would take a group of 10 to grow to a population size of 1000, assuming no limits on growth due to competition for resources.
   The difference equation is $P_{t+1} = (1 + r)P_t = 2.9P_t$, which means that the general solution is $P_t = 10(2.9)^t$.
   The population reaches 1000 when $1000 = 10(2.9)^t \Rightarrow 100 = (2.9)^t \Rightarrow t = \ln(100)/\ln(2.9) \approx$ 4.3 months.

   (c) Now assume that there are limits on growth and that the carrying capacity for a particular environment is is 2000 mice. Write down the difference equation for the logistic model that would describe who the population changes.
   The carrying capacity is 2000 and the intrinsic growth rate is 1.9, so the Logistic model is $P_{t+1} = P_t + 1.9P_t(1 - P_t/2000)$.

4. Suppose a particular model for the per capita growth rate of a population is given by the expression $1 - \frac{P^2}{100}$. Use this equation to find, and simplify a difference equation for this model.

$$\frac{\Delta P}{P} = 1 - \frac{P^2}{100} \Rightarrow \Delta P = P(1 - \frac{P^2}{100})$$

. Thus, the difference equation is

$$P_{t+1} = P_t + \Delta P = P_{t+1} = P_t + P_t(1 - \frac{P_t^2}{100}) = P_t(2 - \frac{P_t^2}{100})$$

5. For the following non-linear model

$$P_{t+1} = 5P_t - P_t^3$$

   (a) Write down an expression for the per capita growth rate.
   First we find

   $$\Delta P = P_{t+1} - P_t = 5P_t - P_t^3 - P_t = 4P_t - P_t^3\,.$$

   Therefore

   $$\frac{\Delta P}{P} = \frac{4P - P^3}{P} = 4 - P^2\,.$$

   (b) What is intrinsic growth rate (ie the per capita growth rate at $P = 0$)?
   The intrinsic growth rate is 4.

   (c) Find the equilibrium population (That is, the population for which the per capita growth rate zero?).
   This happens when $4 - P^2 = 0$, or when $P^2 = 4 \Rightarrow P = 2$.

6. Each of the following difference equations represents logistic growth in a different way. For each model find the intrinsic growth rate and the carrying capacity. (Hint: for the second one you may need to do some manipulation to get it into the right form)

   (a) $P_{t+1} = P_t + 0.1P_t(1 - P_t/10)$
   The general form of the logistic model is $P_{t+1} = P_t + rP_t(1 - P_t/K)$. The given equation is in the right form so we read that the intrinsic growth rate is $r = 0.1$ and the carrying capacity is $K = 10$.

   (b) $P_{t+1} = P_t + 0.2P_t(4 - P_t)$ For this problem we need to put the equation in the right form. Notice that the quantity in the parenthesis is $(4 - P_t)$ is not in the form $1 - \frac{P_t}{K}$. However, we can divide and multiply by 4 to get

   $$(4 - P_t) = \frac{4(4 - P_t)}{4} = 4(1 - \frac{P_t}{4})\,.$$

   Hence, the difference equation is

   $$P_{t+1} = P_t + 0.2P_t(4 - P_t) \Rightarrow P_{t+1} = P_t + 0.2(4)P_t(1 - \frac{P_t}{4}) \Rightarrow P_{t+1} = P_t + 0.8P_t(1 - \frac{P_t}{4})$$

   . Thus $r = 0.8$ and $K = 4$.

7. A tree trunk grows by adding a circular ring with a thickness of 0.5 cm every year.

   (a) Write down a difference equation to describe how the radius, r of a tree at time $t+1$, depends on the radius of the tree at time t.
   $r_{t+1} = r_t + 0.5$

   (b) Find the solution to the difference equation if $r_0 = 2$ cm. That is, find an expression for r_t as a function of time.
   The solution is linear $r_t = 2 + 0.5t$.

   (c) If the cross sectional area of the trunk is denoted by a_t write down an expression for a_t as a function of time.
   $a = \pi r^2$, so $a_t = \pi(2 + 0.5t)^2$. This is a quadratic function.

   (d) Challenge: Find an expression for the non-linear difference equation for a_{t+1} as a function of a_t.