1. Warmups. Given the matrices,

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 2 \\ 4 & -1 \\ 5 & 0 \end{pmatrix}$$

evaluate the following expressions.

(a) AB and BA

$$AB = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -7 \\ 11 & -7 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -11 \\ 2 & -2 \end{pmatrix}$$

(b) B^3

$$B^3 = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 41 & -30 \\ -15 & 11 \end{pmatrix}$$

(c) CD and DC (Hint: The dimensions of the resulting matrices will be different)

$$CD = \begin{pmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & -1 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 22 & 5 \\ 19 & -1 \end{pmatrix}$$

$$DC = \begin{pmatrix} 3 & 2 \\ 4 & -1 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 18 \\ 4 & -5 & 13 \\ 10 & -5 & 20 \end{pmatrix}$$

(d) $A + B$ (We did not define matrix addition in class. What do you think would be a logical way to add matrices?)

$$A + B = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 3 & 2 \end{pmatrix}$$

2. Write each of the following difference equations as a single matrix equation. Then use matrix multiplication to determine the values of x_t and y_t for $t = 1$ and $t = 2$, if $x_0 = 1$ and $y_0 = 2$.

   (a)

$$\begin{aligned} x_{t+1} &= 3x_t + 2y_t \\ y_{t+1} &= x_t + 4y_t \end{aligned}$$

$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = A \begin{pmatrix} x_t \\ y_t \end{pmatrix}$, where $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

Thus $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$ and

$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 39 \\ 43 \end{pmatrix}$.

   (b)

$$\begin{aligned} x_{t+1} &= x_t + y_t \\ y_{t+1} &= 2x_t \end{aligned}$$

$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = A \begin{pmatrix} x_t \\ y_t \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$.

Thus $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and

$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

3. In a particular patch of forest there are only maple and fir trees. Suppose 2% of fir die each year and 4% of maple die each year. Whenever a tree dies a new tree grow in its place. Suppose that 80% of new trees are maple and 20% of new trees are fir. If we let f represent the number of firs and m represent the number of maples, write down a system of difference equations for the growth of trees in the forest. Now express these equations as a single matrix equation.(Challenge: If you have a suitable calculator try computing M^{100} and use this matrix to determine the distribution of trees after 100 years if you start with 500 of each type.)

$$\begin{aligned} m_{t+1} &= 0.96m_t + 0.8(0.04m_t + 0.02f_t) = 0.992m_t + 0.016f_t \\ f_{t+1} &= 0.98f_t + 0.2(0.04m_t + 0.02f_t) = 0.008m_t + 0.984f_t \end{aligned}$$

So the matrix equation is $\begin{pmatrix} m_{t+1} \\ f_{t+1} \end{pmatrix} = \begin{pmatrix} 0.992 & 0.016 \\ 0.008 & 0.984 \end{pmatrix} \begin{pmatrix} m_t \\ f_t \end{pmatrix}$.

4. An annual plant grows from a seed in spring. It produces seeds over the summer and then dies during winter. Some of the seeds germinate in the following spring, some are eaten or rot, and some do not germinate that spring, but stay viable in the soil for an additional year. Suppose a sunflower plant produces 500 seeds. By the spring of each year 95% of sunflower seeds in the environment are eaten or rot, 4% germinate and the other 1% stay in the environment for another year. In addition, suppose only half of sunflower plants that germinate in spring survive to produce seeds at the end of summer. Let S_t be the number of seeds in the environment in January of each year, and F_t be the number of sunflowers at the end of summer of that year. Write difference equations for S_t and F_t.
Here is one answer. $S_{t+1} = 500F_t + 0.01S_t$ and $F_{t+1} = (0.04)(0.5)S_t$

5. Given the matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$,

   (a) Show that multiply the particular vector $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ by A is the same as multiplying the vector by 5. That is, show that $A\vec{v} = 5\vec{v}$;

   $$A\vec{v} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

   (b) challenge: find a vector $\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $A\vec{w} = 2\vec{w}$.

   We want $A\vec{v} = 2\vec{v} \Rightarrow \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$

   $\Rightarrow \begin{pmatrix} 4x + 2y \\ x + 3y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \Rightarrow 4x + 2y = 2x$ and $x + 3y = 2y$.

   Both of these equations imply that $y = -x$. Therefore, any vector of the form $\vec{v} = \begin{pmatrix} x \\ -x \end{pmatrix}$ will work. In particular, taking $x = 1$, $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works.