1. Find the inverse of the matrix $A = \begin{pmatrix} -4 & -1 \\ 5 & 1 \end{pmatrix}$. Check that $A^{-1}A = I$.

   $\text{Det}A = -4(1) - (-1)(5) = -4 + 5 = 1$, so $A^{-1} = \begin{pmatrix} 1 & 1 \\ -5 & -4 \end{pmatrix}$

2. Find the inverse of the matrix $B = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$. Check that $B^{-1}B = I$.

   $\text{Det}A = 2(1) - (3)(4) = -10$, so $A^{-1} = \frac{1}{-10}\begin{pmatrix} 1 & -3 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{pmatrix}$

3. Finding inverses for matrices that are larger than 2×2 is also possible, but harder. However, it is possible to check whether a matrix is an inverse by showing that $A^{-1}A = I$. Show that the inverse of the matrix $A = \begin{pmatrix} 4 & 8 & 3 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} 11 & -12 & -7 \\ -8 & 9 & 5 \\ 7 & -8 & -4 \end{pmatrix}$.

   $$A^{-1}A = \begin{pmatrix} 11 & -12 & -7 \\ -8 & 9 & 5 \\ 7 & -8 & -4 \end{pmatrix}\begin{pmatrix} 4 & 8 & 3 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

4. A matrix is *singular* if it does not have an inverse. This happens when its determinant is zero. Find the values of k such that the following matrix is singular $A = \begin{pmatrix} 4+k & 3 \\ -1 & k \end{pmatrix}$

Det$A = (4+k)(k) - (3)(-1) = 4k + k^2 + 3$. To be singular we want the Det$A = 0$. So $k^2 + 4k + 3 = 0 \Rightarrow (k+3)(k+1) = 0$. So $k = -1$ or $k = -3$. Thus the following matrices are singular $\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}$.

5. The following is the Leslie matrix for a certain animal population divided into two five year classes

$$M = \begin{pmatrix} 2/3 & 3/2 \\ 2/9 & 0 \end{pmatrix}$$

(a) Interpret each of the numbers in the matrix.

2/3 is the birth rate of the first class, 3/2 is the birth rate of the second class, 2/9 is the fraction of the first class that survive into the second class after 5 years. The last term is 0 because no member of the second class survive in the second class after five years.

(b) If there are currently 135 individuals in the first class and 20 individuals in the second class, use your matrix model to predict how many individuals are in each class one five-year period later.

$\vec{P}_0 = \begin{pmatrix} 135 \\ 20 \end{pmatrix}$ and $\vec{P}_1 = M\vec{P}_0 = \begin{pmatrix} 2/3 & 3/2 \\ 2/9 & 0 \end{pmatrix}\begin{pmatrix} 135 \\ 20 \end{pmatrix} = \begin{pmatrix} 120 \\ 30 \end{pmatrix}$. So there will be 120 in the first class and 30 in the second class.

(c) How many were in each class five years ago?

For this we need the inverse matrix, since $\vec{P}_{-1} = M^{-1}\vec{P}_0$.
First Det$M = (2/3)(0) - (2/9)(3/2) = -1/3$. We get

$$M^{-1} = \frac{1}{-1/3}\begin{pmatrix} 0 & -3/2 \\ -2/9 & 2/3 \end{pmatrix} = -3\begin{pmatrix} 0 & -3/2 \\ -2/9 & 2/3 \end{pmatrix} = \begin{pmatrix} 0 & 9/2 \\ 2/3 & -2 \end{pmatrix}.$$

Thus we get

$$\vec{P}_{-1} = \begin{pmatrix} 0 & 9/2 \\ 2/3 & -2 \end{pmatrix}\begin{pmatrix} 135 \\ 20 \end{pmatrix} = \begin{pmatrix} 90 \\ 50 \end{pmatrix}.$$

So there were 90 in the first class and 50 in the second class.

6. The first row of a Leslie matrix shows the birthrates for individuals in each class. The main sub-diagonal of the matrix gives the survival rates of each class (or the rate at which members transition from one class to the next.) The following data ia from page 55 or your text book, for a Leslie model of the US population, for *females* between 0 and 50 years old, divided into 10 equally sized classes. The first row is:

$$(0.000, 0.010, 0.878, 0.3487, 0.4761, 0.3377, 0.1833, 0.0761, 0.0174, 0.0010)$$

The subdiagnoal is:

$$(0.9966, 0.9983, 0.9979, 0.9968, 0.9961, 0.9947, 0.9923, 0.9987, 0.9831)$$

(a) What does the second number in the first row mean? Explain why it is not zero, but the first number is.
This is the birth rate for 5-10 year-olds in the US for a five year period. This is non zero because in that five year period these children age, and some of them pass puberty and have children. The first number is zero because 0-5 year-olds do not age enough in 5 years to have any children.

(b) Why would the first entry in the sub-diagonal be smaller than the second entry? What does this entry mean?
The first entry is the fraction of 0-5 year-olds who reach the second class in five years. Because of infant mortality this survival rate is less than the survival rate for 5-10 year-olds.

(c) Notice that the seventh entry in the sub-diagonal is less than the neighboring values. Can you think of a reason why?
This is is the survival rate for the 30-35 age group. This is the peak child-rearing age and due to the risk of death during child birth this number is slightly lower.

(d) Why is it reasonable to only included females up to 50 in this model?
Very few woman have children after 50, so including those individuals will not have an impact on the growth rate predicted by the model.

7. Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ from last week's workshop

(a) In last week's workshop you were asked to show that multiplying the particular vector $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ by A is the same as multiplying the vector by 5. That is, you showed that $A\vec{v} = 5\vec{v}$. If you haven't done so already check this.

$A\vec{v} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) Now use your result to determine what you think the following matrix products will give. Check your answer by direct computation.

i. $A^2\vec{v}$

We would expect that $A^2 = 5^2\vec{v} = 25\vec{v}$. To check, we first evaluate

$$A^2 = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 18 & 14 \\ 7 & 11 \end{pmatrix}.$$

Thus

$$A^2\vec{v} = \begin{pmatrix} 18 & 14 \\ 7 & 11 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 50 \\ 25 \end{pmatrix} = 25\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5^2\vec{v}\,.$$

ii. $A^4\vec{v}$

In a similar way we expect $A^4 = 5^4\vec{v} = 625\vec{v}$. Checking this we get

$$A^4\vec{v} = \begin{pmatrix} 422 & 406 \\ 203 & 219 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1250 \\ 625 \end{pmatrix} = 625\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 625\vec{v}\,.$$

iii. $A^{-1}\vec{v}$

In a similar way we expect $A^{-1} = 5^{-1}\vec{v} = 0.2\vec{v}$. To check, we first evaluate

$$A^{-1} = \frac{1}{(4)(3) - (2)(1)}\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix}.$$

Thus

$$A^{-1}\vec{v} = \begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \end{pmatrix} = 0.2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0.2\vec{v}\,.$$