1. Given the vector $\vec{P}_0 = \begin{pmatrix} 60 \\ 20 \end{pmatrix}$ find two scalars c_1 and c_2 such that

$$\vec{P}_0 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

We get the two equations

$$\begin{array}{lrcl} (1) & 60 & = & c_1 + 3c_2 \\ (2) & 20 & = & c_1 - c_2 \end{array}$$

So we take equation (1) minus equation (2) to eliminate c_1. This gives us

$$60 - 20 = 3c_2 - (-c_2) \;\Rightarrow\; 40 = 4c_2 \;\Rightarrow c_2 = 10 \, .$$

Substituting into equation (2) gives $20 = c_1 - 10 \;\Rightarrow\; c_1 = 30$.

2. The following is the Leslie matrix for a certain animal population divided into two five year classes

$$M = \begin{pmatrix} 2/3 & 3/2 \\ 2/9 & 0 \end{pmatrix}$$

(a) Find the eigenvalues λ_1 and λ_2 for this matrix given that the eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 9 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$M\vec{v} = \lambda\vec{v}$ if $\vec{v}$ is an eigenvector, by definition. But

$$M\vec{v}_1 = \begin{pmatrix} 2/3 & 3/2 \\ 2/9 & 0 \end{pmatrix} \begin{pmatrix} 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix} .$$

Therefore $\lambda_1 = 1$. Similarly

$$M\vec{v}_2 = \begin{pmatrix} 2/3 & 3/2 \\ 2/9 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2/3 \end{pmatrix} = (-1/3) \begin{pmatrix} -3 \\ 2 \end{pmatrix} ,$$

so $\lambda_2 = -1/3$.

(b) If the initial population of the two classes is

$$\vec{P}_o = \begin{pmatrix} 30 \\ 60 \end{pmatrix}$$

find the scalars c_1 and c_2 such that

$$\vec{P}_o = c_1\vec{v}_1 + c_2\vec{v}_2$$

We get the two equations

$$\begin{array}{lrcl} (1) & 30 & = & 9c_1 - 3c_2 \\ (2) & 60 & = & 2c_1 + 2c_2 \end{array}$$

If we take equation $2\times$ eqn(1) plus $3\times$ eqn(2) we can eliminate c_2. This gives us

$$60 + 180 = 18c_1 + 6c_1 - 6c_2 + 6c_2 \ \Rightarrow\ 240 = 24c_1 \ \Rightarrow c_1 = 10\ .$$

Substituting into equation (2) gives $60 = 2(10) + 2c_2 \ \Rightarrow\ c_2 = 40/2 = 20.$

(c) Write down the general solution for $\vec{P}_t$ in terms of eigenvectors and eigenvalues.

$$\vec{P}_t = 10(1)^t \begin{pmatrix} 9 \\ 2 \end{pmatrix} + 20(-1/3)^t \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

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(d) Describe the long term behavior of the population. That is, describe the dominant solution, and the stable stage distribution. Sketch a graph of each population group as a function of time, clearly showing the transient behavior on your graph.
The dominant eigenvalue is $\lambda_1 = 1$ and $(1)^t = 1$, so the dominant solution is an equilibrium equal to $10\begin{pmatrix} 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 90 \\ 20 \end{pmatrix}$. The stable stage distribution is 9:2. The other eigenvalue is $(-1/3)$ so the transient behavior is an oscillating decay. $x_t = 90 - 60(-1/3)^t$ and $y_t = 20 + 40(-1/3)^t$. So x_t starts at $x_0 = 90$ and then oscillates above and below $x = 90$, gradually approaching this equilibrium value. Similarly $y_t =$ starts at $y_0 = 60$ and oscillates below and then above $y = 20$, gradually approaching this equilibrium value.

3. In this example you will learn how to find the eigenvalues and eigenvectors of a matrix. Consider the matrix $M = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$. We want to find λ and $\vec{v}$ such that $M\vec{v} = \lambda\vec{v}$. This means that

$$\begin{aligned} M\vec{v} - \lambda\vec{v} &= 0 \\ M\vec{v} - \lambda I\vec{v} &= 0 \\ (M - \lambda I)\vec{v} &= 0 \end{aligned}$$

where we introduce the identity matrix I so that we can factor out the vector $\vec{v}$. Since we assume that $\vec{v}$ is not zero, this means that the matrix $M - \lambda I$ must be singular. That is $\mathrm{Det}(M - \lambda I) = 0$.

(a) Show that $M - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$

$$M - \lambda I = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}.$$

(b) Evaluate $\mathrm{Det}(M - \lambda I)$.
$\mathrm{Det}(M - \lambda I) = (4-\lambda)(3-\lambda) - (2)(1) = 12 - 7\lambda + \lambda^2 - 2 = \lambda^2 - 7\lambda + 10.$

(c) Using the result above, solve the equation $\mathrm{Det}(M - \lambda I) = 0$, for λ.
$\mathrm{Det}(M - \lambda I) = 0 \;\Rightarrow\; \lambda^2 - 7\lambda + 10 = 0 \;\Rightarrow\; (\lambda - 5)(\lambda - 2) = 0 \;\Rightarrow \lambda = 5 \quad \text{or} \quad \lambda = 2.$

(d) You should have two values for λ. For each eigenvalue find the corresponding eigenvector by finding a vector that satisfies the equation $\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}\vec{v} = 0.$

First we take $\lambda = 5$.

$$\begin{pmatrix} 4-5 & 2 \\ 1 & 3-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \;\Rightarrow\; -x + 2y = 0 \quad \text{or} \quad x - 2y = 0 \;\Rightarrow\; x = 2y\ .$$

So $\vec{v} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for any non-zero y (easiest to take $y = 1$). Now we take $\lambda = 2$.

$$\begin{pmatrix} 4-2 & 2 \\ 1 & 3-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \;\Rightarrow\; 2x + 2y = 0 \quad \text{or} \quad x + y = 0 \;\Rightarrow\; x = -y\ .$$

So $\vec{v} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for any non-zero y.

4. The weather in Olympia is either wet or dry. If the weather in Olympia is wet on one day then there is 80% chance that the weather will be wet the next day. If the weather is dry one day then their is 40% chance it will be dry the next day.Let w_t be the probability it is wet today and d_t be the probability it is dry on a particular day.

   (a) Write down an expression for the probability that it is wet the next day w_{t+1} in terms of w_t and d_t.
   $w_{t+1} = 0.8w_t + 0.6d_t$

   (b) Write down an expression for the probability that it is dry the next day d_{t+1} in terms of w_t and d_t.
   $d_{t+1} = 0.2w_t + 0.4d_t$

   (c) Write the two expressions above as a matrix equation $\vec{P}_{t+1} = M\vec{P}_t$, where $\vec{P}_t = \begin{pmatrix} w_t \\ d_t \end{pmatrix}$

   $$M = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$$

   (d) Use the method from the previous question to find the eigenvalues and eigenvectors of M.
   First the eigenvalues: Det$M - \lambda I = (0.8 - \lambda)(0.4 - \lambda) - (0.6)(0.2) = \lambda^2 - 1.2\lambda + 0.2 = (\lambda - 1)(\lambda - .2)$. So the eigenvalues are $\lambda = 1$ and $\lambda = 0.2$. For $\lambda = 1$ the eigenvector can be found from solving

   $$\begin{pmatrix} 0.8 - 1 & 0.6 \\ 0.2 & 0.4 - 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow -0.2x + 0.6y = 0 \Rightarrow x = 3y \Rightarrow \vec{v} = \begin{pmatrix} 3y \\ y \end{pmatrix} = y\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

   For $\lambda = 0.2$ we find

   $$\begin{pmatrix} 0.8 - 0.2 & 0.6 \\ 0.2 & 0.4 - 0.2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow 0.6x + 0.6y = 0 \Rightarrow x = -y \Rightarrow \vec{v} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

   (e) Use these values to determine the probability of it being wet and dry in Olympia far into the future.

   The dominant eigenvalue is $\lambda = 1$ with eigenvector $\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. So the probability of it being wet or dry reaches an equilibrium with the ratio of wet to dry being 3 to 1. That is, in the long term there is a 75% chance of it raining and a 25% chance of it being dry.