1. Suppose two types of individuals A and B occur in the population with relative frequency x and y, and with fitness f_A and f_B respectively, so that the replicator equations are

$$\begin{aligned} \Delta x &= x(f_A - \phi) \\ \Delta y &= y(f_B - \phi) \end{aligned}$$

where ϕ is a "selection" function, to be determined. Prove that in order to maintain $x+y=1$ the function ϕ must take the form $\phi = f_A x + f_B y$. (Hint: Try adding the two equations above and use the fact that if $x+y=1$ then $\Delta x + \Delta y = 0$).
If we add the two equations we get

$$\begin{aligned} \Delta x + \Delta y &= x(f_A - \phi) + y(f_B - \phi) \\ &= x f_A - x\phi + y f_B - y\phi \\ &= f_A x + f_B y - (x+y)\phi \end{aligned}$$

We then make use of the facts that $x+y=1$ and $\Delta x + \Delta y = 0$ to get

$$0 = f_A x + f_B y - \phi \;\Rightarrow\; \phi = f_A x + f_B y$$

2. The above equations represent both reproduction (f_A and f_B) and selection (ϕ). Now suppose that A mutates to B with probability α and B mutates to A with probability β.

   (a) Explain why the replicator equations become

$$\begin{aligned} \Delta x &= x(f_A - \phi) - \alpha x + \beta y \\ \Delta y &= y(f_B - \phi) + \alpha x - \beta y \end{aligned}$$

   (b) Using the fact $y = 1 - x$ show that the above system reduces to the single equation

$$\Delta x = (f_A - f_B)x(1-x) - (\alpha+\beta)x + \beta$$

   First, $\phi = f_A x + f_B y = f_A x + f_B(1-x) = f_B + x(f_A - f_B)$. Then inserting this into the first equation we get

$$\begin{aligned} \Delta x &= x(f_A - (f_B + x(f_A - f_B))) - \alpha x + \beta(1-x) \\ \Delta x &= x(f_A - f_B - x(f_A - f_B)) - \alpha x + \beta - \beta x \\ \Delta x &= x(f_A - f_B)(1-x) - (\alpha+\beta)x + \beta \end{aligned}$$

   (c) Suppose $f_A = f_B + 1$. Without mutation this would lead to the extinction of B because A is fitter. Suppose A can mutate to B but B does not mutate to A. Show that A and B can then coexist with equilibrium distribution $x = 1 - \alpha$.
   $\beta = 0$ and $f_A - f_B = 1$ so $\Delta x = x(1-x) - \alpha x = x(1-\alpha-x)$. Equilibrium is when $\Delta x = 0$ which is when $x = 0$ or $1 - \alpha - x = 0$. The second equation implies $x = 1 - \alpha$.

3. Suppose we have an evolutionary game between individuals of type A and type B with reward matrix.

	A	B
A	a	b
B	c	d

In game theory, A and B are called strategies because they often correspond to types of behavior that individuals might choose to take when they interact (eg fight or not fight, share or not share etc). An evolutionarily stable strategy (ESS) is one which will oppose the invasion of a rare mutant strategy. Suppose in a large population of type A individuals a rare mutant of type B is introduced. Then A will be an ESS if one of the two situations arises: (i) $a > c$ or (ii) $a = c$ and $b > d$. When B is rare it will most likely meet another A. If condition (i) holds B will be less fit then A and die out. If condition (ii) holds B will have the same fitness as an A when they interact with other A's. However, in this case the rare circumstances when they both meet another B will be important. Then the second part of condition (ii) will guarantee that B is less fit than A from those interactions. For each of the following matrices, decide whether one, both or none of the strategies are ESS, and whether any of the games allow both strategies to coexist.

(a)

	A	B
A	2	1
B	3	5

(b)

	A	B
A	5	3
B	5	2

(c)

	A	B
A	5	1
B	2	3

(d)

	A	B
A	2	5
B	3	1

For (a) A is not ESS because $2 < 3$, but B is ESS, because $5 > 1$. For (b) A is ESS, because although the first column has identical entries, the second column has $3 > 2$. B is not ESS. For (c) A is ESS because $5 > 2$, and B is also ESS because $3 > 1$. This is a bistable situation. For (d) neither is ESS, but there is a stable coexistence.

4. For which values of the parameter p does the following game have a stable coexistence of A and B? Find the equilibrium value of the population of A individuals as a function of p. For which value of p will the population be equally distributed between A and B individuals?

	A	B
A	2	5
B	p	0

For equilibrium to be possible we need B to be fittest when A is common and A to be fittest when B is common. Since $5 > 0$, the second condition is satisfied, Thus we need $p > 2$ to satisfy the first condition. Equilibrium is when the fitness of A and B are equal. Hence $f_A = f_B \Rightarrow 2x + 5y = px + 0$. If we put $y = 1 - x$ we get:
$2x + 5(1 - x) = px \Rightarrow -3x + 5 = px \Rightarrow 5 = (p + 3)x \Rightarrow x = 5/(p + 3)$. Equal distribution is when $x = \frac{1}{2}$, which means $p + 3 = 10$. Thus $p = 7$.

5. In the Hawk-Dove game two individuals compete for a resource with a fitness benefit b (perhaps food, a home, or a mate). An individual using the Hawk strategy postures first but escalates to a fight if threatened. An individual using the Dove strategy postures first, but retreats if threatened. Individuals who fight and lose incur a cost c and get no benefit. Those who fight and win incur no cost and get benefit b. When a individual using the Hawk strategy meets another Hawk imagine that half the time they lose and half the time they win. Similarly, when two Dove players meet imagine that half the time they get the benefit and half the time they don't. Then a good matrix describing this game is:

	H	D
H	$\frac{1}{2}(b-c)$	b
D	0	$\frac{1}{2}b$

(a) Assuming b and c are both bigger than zero can Dove ever be an ESS?
No. For Dove to be an ESS we need $\frac{1}{2}b \geq b$. This can only happen if $b \leq 0$ which does not make sense for this game. A negative benefit is not a benefit!

(b) What is the condition on the cost c for Hawk to be an ESS?
Here we need $\frac{1}{2}(b-c) \geq 0$ which means $c \leq b$. So if the cost of fighting is less than the benefit of the resource it always pays to fight!

(c) What happens to the population of Hawks and Doves if c does not satisfy this condition?
If $c > b$ then Doves can invade a Hawk only strategy. However, Doves do not take over since they are not an ESS. Instead there is a stable equilibrium of Hawks and Doves.

(d) What are the values of b and c corresponding to the Hawk-Dove reward matrix below?

	H	D
H	-2	2
D	0	1

By direct comparison $b = 2$ and $\frac{1}{2}(b-c) = -2$ which means $2 - c = -4 \Rightarrow c = 6$.

(e) Show Hawk and Dove can coexist and find the expected equilibrium distribution of Hawks and Doves.
Since $c > b$ we have the condition stated in part (c) for an equilibrium. This occurs when $f_H = f_D \Rightarrow -2x + 2y = 0x + y \Rightarrow y = 2x \Rightarrow 1 - x = 2x \Rightarrow 1 = 3x \Rightarrow x = \frac{1}{3}$. So a population where one third play a Hawk-like strategy and two thirds play a Dove-like strategy is stable.