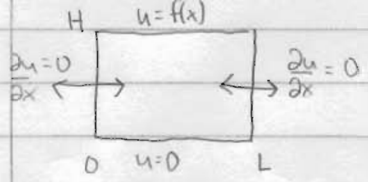
PDE Homework Solutions Week 4

2.5.1.a B.C. $\frac{\partial u}{\partial x}(0,y)=0$ $\quad \frac{\partial u}{\partial x}(L,y)=0$ $\quad u(x,0)=0$ $\quad u(x,H)=f(x)$



$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

let $u = X(x)Y(y)$

$$\nabla^2 u = Y(y)X''(x) + Y''(y)X(x) = 0 \Rightarrow X''(x)Y(y) = -X(x)Y''(y)$$

$$\Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} = \lambda$$

consider $X'' = -\lambda X$

B.C. $\frac{\partial u}{\partial x}(0,y) = 0 \Rightarrow X'(0)Y(y) = 0$

$\Rightarrow X'(0) = 0$

$\frac{\partial u}{\partial x}(L,y) = 0 \Rightarrow X'(L) = 0$

$X = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$

$X' = -\sqrt{\lambda}a\sin\sqrt{\lambda}x + \sqrt{\lambda}b\cos\sqrt{\lambda}x$

$X'(0) = 0 + \sqrt{\lambda}b\cos\sqrt{\lambda}x = 0 \Rightarrow b = 0$

$X'(L) = -\sqrt{\lambda}a\sin\sqrt{\lambda}L = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$

$\Rightarrow X = a\cos\left(\frac{n\pi x}{L}\right)$ $n = 0,1,2\ldots$

consider $Y'' = \lambda Y$

$\Rightarrow Y = Ae^{\pm\sqrt{\lambda}y}$

$\Rightarrow Y = ae^{\sqrt{\lambda}y} + be^{-\sqrt{\lambda}y}$

B.C. $u(x,0) \Rightarrow X(x)Y(0) = 0$

$\Rightarrow Y(0) = 0$

$Y(0) = ae^0 + be^0 = 0$

$\Rightarrow a + b = 0 \Rightarrow b = -a$

$Y(y) = ae^{\sqrt{\lambda}y} - ae^{-\sqrt{\lambda}y} = \frac{2a(e^{\sqrt{\lambda}y} - e^{-\sqrt{\lambda}y})}{2}$

$\Rightarrow Y = 2a\sinh(\sqrt{\lambda}y) = 2a\sinh\left(\frac{n\pi y}{L}\right)$ $n = 0,1,2\ldots$

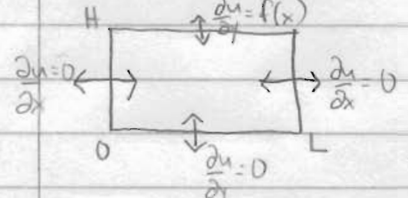
Combining X and Y $\Rightarrow u(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$

Notice that when $y=0$, $\sinh\left(\frac{n\pi y}{L}\right) = 0$ and $u(x,y) = 0$. To preserve this trait, the constant A_0 outside the sum must be multiplied by y.

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

where $A_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ from a formula derived in class

2.5.2 (a) considering a rectangle with B.C. $\frac{\partial u}{\partial x}(0,y)=0$ $\quad \frac{\partial u}{\partial y}(x,0)=0$



$\frac{\partial u}{\partial x}(L,y)=0 \quad \frac{\partial u}{\partial y}(x,H)=f(x)$

The heat flow given for the top boundary is non-homogeneous, but the average heat flow for the top boundary must be zero in order for a solution to exist.
We expect $\int_0^L f(x)\,dx=0$.

(b) from seperation of variables
$u(x,y)=X(x)Y(y) \Rightarrow X''=-\lambda X$ and $Y''=\lambda Y$

Consider $X''=-\lambda X$
$\Rightarrow X=a\cos\sqrt{\lambda}x+b\sin\sqrt{\lambda}x$
$X'=-\sqrt{\lambda}a\sin\sqrt{\lambda}x+\sqrt{\lambda}b\cos\sqrt{\lambda}x$
$\frac{\partial u}{\partial x}(0,y) \Rightarrow X'(0)Y(y)=0 \Rightarrow X'(0)=0$
$\frac{\partial u}{\partial x}(L,y) \qquad \Rightarrow X'(L)=0$
$X'(0)=0+\sqrt{\lambda}b\cos\sqrt{\lambda}x=0 \Rightarrow b=0$
$X'(L)=-\sqrt{\lambda}a\sin\sqrt{\lambda}x=0 \Rightarrow \lambda=\left(\frac{n\pi}{L}\right)^2$
$X=a\cos\left(\frac{n\pi x}{L}\right) \quad n=0,1,2\ldots$

consider $Y''=\lambda Y$
$\Rightarrow Y=ae^{\sqrt{\lambda}y}+be^{-\sqrt{\lambda}y}$
$Y'=\sqrt{\lambda}ae^{\sqrt{\lambda}y}-\sqrt{\lambda}be^{-\sqrt{\lambda}y}$
$\frac{\partial u}{\partial y}(x,0)=0 \Rightarrow X(x)Y'(0)=0 \Rightarrow Y'(0)=0$
$Y'(0)=\sqrt{\lambda}ae^0-\sqrt{\lambda}be^0=0$
$\Rightarrow a-b=0 \Rightarrow a=b$
$Y(y)=ae^{\sqrt{\lambda}y}+ae^{-\sqrt{\lambda}y}=2a\underline{\left(e^{\sqrt{\lambda}y}+e^{-\sqrt{\lambda}y}\right)}\over 2$
$\Rightarrow Y=2a\cosh\left(\frac{n\pi y}{L}\right)$

and...

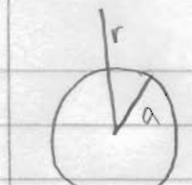
$$\boxed{u(x,y)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right)} \to \text{now apply } \frac{\partial u}{\partial y}(x,H)=f(x)$$

$$\frac{\partial u}{\partial y}=0+\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)A_n\cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)=f(x)$$

if we expand $f(x)$ into a fourier cosine series we'd expect $A_0=a_0=0$
(similarly $A_n=a_n$, but we won't show that.)
$a_0=\int_0^L f(x)\,dx=0$ which is what we predicted in part (a)

(c) This leaves A_0 in our solution as an arbitrary constant. We're asked to determine it using the time-dependent function $u(x,y,0)=g(x,y)$. We can use this to find the average temperature over the area of the rectangle when $t=0$. Use integration to find the average:

$$A_0=\frac{1}{HL}\int_0^L\int_0^H g(x,y)\,dy\,dx$$

2.5.3  $r \geq a$

in polar coordinates $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

We need periodicity and boundedness conditions: $u(r,-\pi) = u(r,\pi)$

$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,-\pi)$

let $u(r,\theta) = R(r)\Theta(\theta)$ $\qquad |u(r,\theta)| < \infty$

$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\Theta + \frac{1}{r^2}R\frac{d^2\Theta}{d\theta^2} = 0 \Rightarrow \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{-1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \lambda$

As usual, $\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta \Rightarrow \Theta = a\cos\sqrt{\lambda}\theta + b\sin\sqrt{\lambda}\theta$

and from the periodicity conditions $\Theta(\pi) = \Theta(-\pi)$

$\Rightarrow a\cos\sqrt{\lambda}\pi + b\sin\sqrt{\lambda}\pi = a\cos(-\sqrt{\lambda}\pi) + b\sin(-\sqrt{\lambda}\pi)$

using a trig identity $\Rightarrow a\cos\sqrt{\lambda}\pi + b\sin\sqrt{\lambda}\pi = a\cos\sqrt{\lambda}\pi - b\sin\sqrt{\lambda}\pi$

Simplifying $\Rightarrow 2b\sin\sqrt{\lambda}\pi = 0 \Rightarrow \lambda = n^2$

Now consider $r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \lambda R = n^2 R$ $\qquad$ guess $R = r^p$ $n \neq 0$

$\Rightarrow \frac{dR}{dr} = pr^{p-1}$ and $r\frac{dR}{dr} = pr^p$

$\Rightarrow \frac{d}{dr}(pr^p) = p^2 r^{p-1}$

$\Rightarrow r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = p^2 r^p$

$p^2 r^p = n^2 R = n^2 r^p$

$\Rightarrow p = \pm n$ so...

$R(r) = cr^n + dr^{-n}$ $\quad n \neq 0$ $\qquad$ $n = 0 \Rightarrow \cancel{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$ and $r\frac{dR}{dr} = \text{constant} = c_1$

and $R(r) = c_1 \ln r + c_2$ $\quad n = 0$ $\qquad$ $\Rightarrow \frac{dR}{dr} = \frac{c_1}{r} \Rightarrow R = c_1 \ln r + c^2$

but $|u(r,\theta)| < \infty$

as $r \to \infty$ (outside the disk) cr^n and $c_1 \ln r$ blow up

$\Rightarrow R = dr^{-n}$ $n \neq 0$; $R = c_2 = d$ $n = 0$ (d combines with a and b into A_0, A_n, B_n)

so...

$$\boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{-n}\cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n}\sin(n\theta)}$$

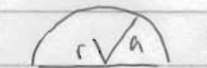
now apply the boundary conditions...

(a) we can identify A_0, A_n and B_n since this B.C. is so "nice"

$u(a,\theta) = \ln 2 + 4\cos 3\theta \Rightarrow A_0 = \ln 2$ $\; A_3 = 4$ $\; A_n = 0$ $n \neq 0,3$ $\; B_n = 0$

(b) Given $u(a,\theta) = f\theta \Rightarrow A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$ $\quad A_n = \frac{1}{a^{-n}\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$

and $B_n = \frac{1}{a^{-n}\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$ from pg. 30 (2.5.47)

2.5.6 

Similar to problem 2.5.3, $\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{-1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \lambda$

$\Rightarrow \Theta = a\cos(\sqrt{\lambda}\,\theta) + b\sin(\sqrt{\lambda}\,\theta)$ and $R = cr^n + dr^{-n} \quad n \neq 0$

$R = c_1 \ln r + c_2 \quad n = 0$

but $|u(r,\theta)| < \infty \Rightarrow d = 0,\ c_1 = 0$

(as $r \to -\infty$ r^{-n} and $\ln r$ blow up)

$R = cr^n$

(a) $u = 0$ on the diameter and $u(a,\theta) = g(\theta)$

$\Rightarrow \Theta(0) = 0$ and $\Theta(\pi) = 0$

$\Theta(0) = a\cos(0) + b\sin(0) = a = 0 \Rightarrow a = 0$

$\Theta(\pi) = b\sin(\sqrt{\lambda}\,\pi) = 0 \Rightarrow \lambda = n^2$

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} B_n r^n \sin(n\pi) \quad \text{and} \quad B_n = \frac{1}{a^n\pi}\int_0^{\pi} g(\theta)\sin(n\theta)\,d\theta}$$

(b) diameter is insulated $\Rightarrow \frac{\partial u}{\partial\theta}(0) = 0$ and $\frac{\partial u}{\partial\theta}(\pi) = 0$

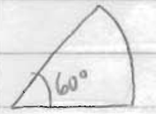
$\Theta = a\cos(\sqrt{\lambda}\,\theta) + b\sin(\sqrt{\lambda}\,\theta)$

$\Theta' = -\sqrt{\lambda}\,a\sin(\sqrt{\lambda}\,\theta) + \sqrt{\lambda}\,b\cos(\sqrt{\lambda}\,\theta)$

$\Theta'(0) = 0 + \sqrt{\lambda}\,b = 0 \Rightarrow b = 0$

$\Theta'(\pi) = -\sqrt{\lambda}\,a\sin(\sqrt{\lambda}\,\pi) = 0 \Rightarrow \lambda = n^2$

$$\boxed{\begin{aligned} u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) \quad \text{and} \quad A_0 &= \frac{1}{2\pi}\int_0^{\pi} g(\theta)\cos(n\theta)\,d\theta \\ A_n &= \frac{1}{a^n\pi}\int_0^{\pi} g(\theta)\cos(n\theta)\,d\theta \end{aligned}}$$

2.5.7.b 

$0 \le \theta \le \frac{\pi}{3}$ $\quad \frac{\partial u}{\partial \theta}(r,0)=0 \quad \frac{\partial u}{\partial \theta}(r,\frac{\pi}{3})=0 \quad u(a,\theta)=f(\theta)$

as before $\Theta = a\cos\sqrt{\lambda}\,\theta + b\sin\sqrt{\lambda}\,\theta$

$\Theta' = -\sqrt{\lambda}\,a\sin\sqrt{\lambda}\,\theta + \sqrt{\lambda}\,b\cos\sqrt{\lambda}\,\theta \qquad \Theta'(0)=0 \Rightarrow b=0$

$\Theta'(\frac{\pi}{3})=0 \Rightarrow \sin(\sqrt{\lambda}\frac{\pi}{3})=0$

so $\Theta = a\cos(3n\theta)$ $\qquad \Rightarrow \sqrt{\lambda}=3n$

$\Rightarrow \lambda = 9n^2$

and $r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \lambda R = 9n^2 R$

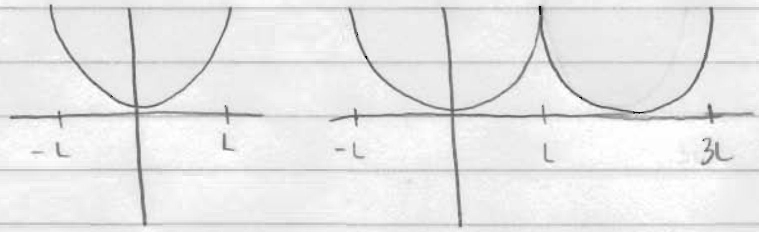
$\Rightarrow R = cr^{3n} + dr^{-3n} \quad n \neq 0$

$R = c_1 \ln r + c_2 \quad n=0$

$|u(r,\theta)| < \infty \Rightarrow d, c_1 = 0$

so... $\boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{3n}\cos(3n\theta) \qquad A_0 = \frac{1}{2\pi}\int_0^{\pi/3} f(\theta)\,d\theta \qquad A_n = \frac{1}{\pi a^{3n}}\int_0^{\pi/3} f(\theta)\cos(3n\theta)\,d\theta}$

3.2.1.b $f(x) = x^2$



$f(x)$ $\qquad$ Fourier series of $f(x)$

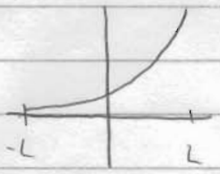
3.2.2.f $f(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{0} 0\,dx + \frac{1}{2L}\int_0^L 1\,dx = \frac{1}{2L}\left[x\right]_0^L = \frac{1}{2}$

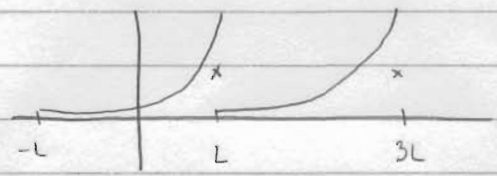
$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx = \frac{1}{L}\int_{-L}^{0} 0\,dx + \frac{1}{L}\int_0^L \cos(\frac{n\pi x}{L})\,dx = \frac{1}{L}\left[\frac{-n\pi}{L}\sin(\frac{n\pi x}{L})\right]_0^L = 0$

$b_n = \frac{1}{L}\int_{-L}^{0} 0\,dx + \frac{1}{L}\int_0^L \sin(\frac{n\pi x}{L})\,dx = \frac{1}{L}\left[\frac{n\pi}{L}\cos(\frac{n\pi x}{L})\right]_0^L = \frac{1}{n\pi}\left(1-\cos(n\pi)\right) = \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

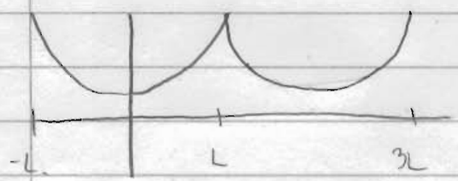
3.3.1.d $f(x) = e^x$



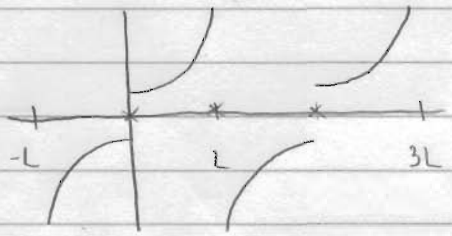
f(x)



fourier series of f(x)



fourier cosine series



fourier sine series

3.3.7 $e^x = f(x)$ $\quad f_{even}(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $f_{odd}(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$

$$f(x) = f_{even}(x) + f_{odd}(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x} + \frac{1}{2}e^x - \frac{1}{2}e^{-x} = e^x$$

3.3.10 $f(x) = \begin{cases} x^2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$ $\quad f_e(x) = \begin{cases} \frac{1}{2}\left(x^2 + e^x\right) & x < 0 \\ \frac{1}{2}\left(x^2 + e^{-x}\right) & x > 0 \end{cases}$ $\quad f_o(x) = \begin{cases} \frac{1}{2}\left(x^2 - e^{-x}\right) & x < 0 \\ \frac{1}{2}\left(e^{-x} - x^2\right) & x > 0 \end{cases}$