1. The equation for the distance between two points on the surface of a unit sphere, along a path specified by $\theta = \theta(\phi)$ where θ the polar or zenith angle (down from the pole), and ϕ measures the azimuthal angle (around the polar axis), is given by the functional

$$I[\theta] = \int_{\phi_1}^{\phi_2} \sqrt{(\phi')^2 + \sin^2\theta}\, d\phi$$

Find a second order differential equation for the geodesic (shortest path between two points).

2. An object follows a path parameterized in plane polar coordinates by $r(t)$ and θt which minimizes the following functional.

$$\int \left(\tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{r} \right) dt$$

Write down Euler lagrange equations for the two dependent variables, and then by finding the first integrals, find a system of first order differential equations describing the motion of the object.

1. Suppose you are given an integral expression that is to be optimized

$$I[y] = \int_a^b F(x, y, y', y'')dx$$

where F depends explicitly on the function $y = y(x)$ and its first two derivatives y' and y''. Show that y must satisfy a modified Euler-Lagrange equation given by

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

and state what conditions the variation $\eta(x)$ must satisfy at $x = a$ and $x = b$ in order for this expression to be valid.

2. According to Einstein's theory of gravity the path of light is influenced by massive bodies. In the case of a light ray travelling in the equatorial plane of a spherically symmetric star the light follows a path in polar coordinates $r = f(\theta)$ which minimizes the integral

$$\int \frac{r\sqrt{(\frac{dr}{d\theta})^2 + r(r - 2\alpha)}}{r - 2\alpha} d\theta$$

where $\alpha = GM/c^2$, where G is the gravitational constant, c is the speed of light in a vacuum and M is the mass of the star.

Show that differential equation describing the path of light near the star can be expressed as

$$\frac{d^2u}{d\theta^2} + u = 3\alpha u^2$$

where $u = 1/r$. Explain why the non-linear term on the right hand side of the equation is negligible if the mass of the star is small, or if the path of light is a long way from the star. Solve the differential equation in this case and describe the path of the light ray

3. Solve the following problems from *Perfect Form* by Lemons: 3.4 and 3.6