11 Partial Derivatives

11.1 Functions of Several Variables

1. (a) From Table 1, $f(8, 60) = -7$, which means that if the temperature is $8\,^\circ\text{C}$ and the wind speed is 60 km/h, then the air would feel equivalent to approximately $-7\,^\circ\text{C}$ without wind.

(b) The question is asking: when the temperature is $-12\,^\circ\text{C}$, what wind speed gives a wind-chill index of $-26\,^\circ\text{C}$? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 80 km/h, what temperature gives a wind-chill index of $-14\,^\circ\text{C}$? From Table 1, the temperature is $4\,^\circ\text{C}$.

(d) The function $I = f(-4, v)$ means that we fix T at -4 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-4\,^\circ\text{C}$. From Table 1 (look at the row corresponding to $T = -4$), the function decreases and appears to approach a constant value as v increases.

(e) The function $I = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h. From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

2. (a) From the table, $f(95, 70) = 124$, which means that when the actual temperature is $95\,^\circ\text{F}$ and the relative humidity is 70%, the perceived air temperature is approximately $124\,^\circ\text{F}$.

(b) Looking at the row corresponding to $T = 90$, we see that $f(90, h) = 100$ when $h = 60$.

(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 88$ when $T = 85$.

(d) $I = f(80, h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is $80\,^\circ\text{F}$. Similarly, $I = f(100, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is $100\,^\circ\text{F}$. Looking at the rows of the table corresponding to $T = 80$ and $T = 100$, we see that $f(80, h)$ increases at a relatively constant rate of approximately $1\,^\circ\text{F}$ per 10% relative humidity, while $f(100, h)$ increases more quickly (at first with an average rate of change of $5\,^\circ\text{F}$ per 10% relative humidity) and at an increasing rate (approximately $12\,^\circ\text{F}$ per 10% relative humidity for larger values of h).

3. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$\begin{aligned} P(2L, 2K) &= 1.01(2L)^{0.75}(2K)^{0.25} = 1.01\left(2^{0.75}\right)\left(2^{0.25}\right)L^{0.75}K^{0.25} = \left(2^1\right)1.01L^{0.75}K^{0.25} \\ &= 2P(L, K) \end{aligned}$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^\alpha K^{1-\alpha}$:
$P(2L, 2K) = b(2L)^\alpha(2K)^{1-\alpha} = b(2^\alpha)(2^{1-\alpha})\,L^\alpha K^{1-\alpha} = (2^{\alpha+1-\alpha})bL^\alpha K^{1-\alpha} = 2P(L, K)$.

8. (a) $g(2,-2,4) = \ln\left(25 - 2^2 - (-2)^2 - 4^2\right) = \ln 1 = 0.$

(b) For the logarithmic function to be defined, we need $25 - x^2 - y^2 - z^2 > 0$. Thus the domain of g is $\{(x,y,z) \mid x^2 + y^2 + z^2 < 25\}$, the interior of the sphere $x^2 + y^2 + z^2 = 25$.

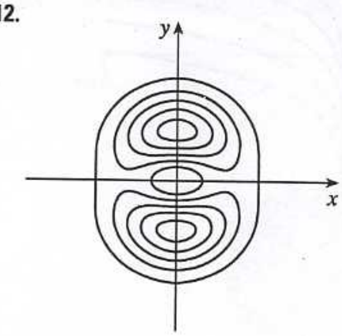
(c) Since $0 < 25 - x^2 - y^2 - z^2 \leq 25$ for (x,y,z) in the domain of g, $\ln\left(25 - x^2 - y^2 - z^2\right) \leq \ln 25$. Thus the range of g is $(-\infty, \ln 25]$.

9. The point $(-3,3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3,3) \approx 56$. The point $(3,-2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3,-2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

10. If we start at the origin and move along the x-axis, for example, the z-values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z-values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

11. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

12.



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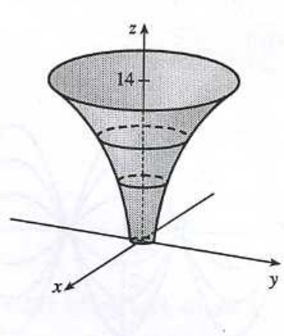

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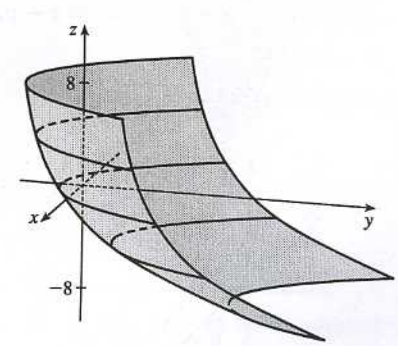
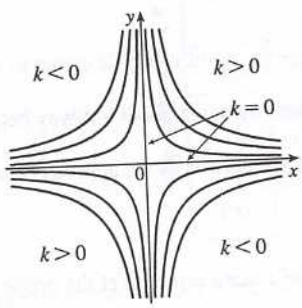
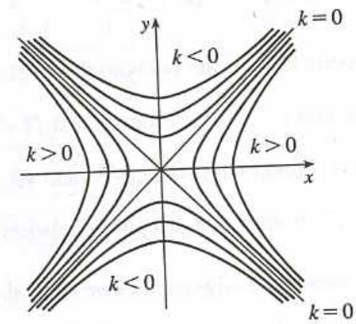


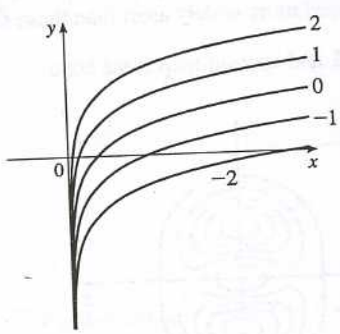
15. The level curves are $xy = k$. For $k = 0$ the curves are the coordinate axis; if $k > 0$, they are hyperbolas in the first and third quadrants; if $k < 0$, they are hyperbolas in the second and fourth quadrants.



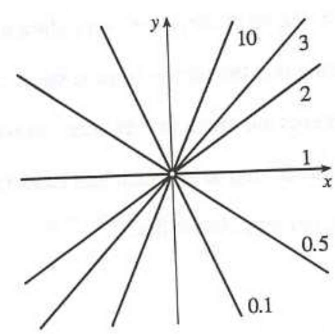
16. The level curves are $k = x^2 - y^2$. When $k = 0$, these are the lines $y = \pm x$. When $k > 0$, the curves are hyperbolas with axis the x-axis and when $k < 0$, they are hyperbolas with axis the y-axis.



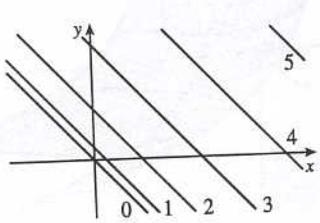
17. The level curves are $y - \ln x = k$ or $y = \ln x + k$.



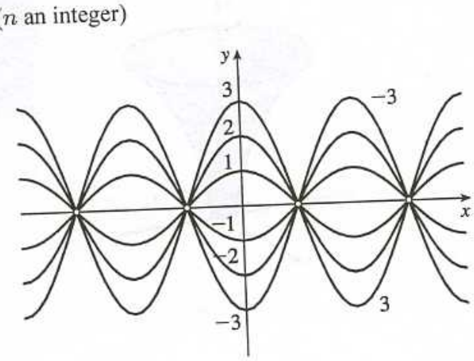
18. The level curves are $e^{y/x} = k$ or equivalently $y = x \ln k$ $(x \neq 0)$, a family of lines with slope $\ln k$ $(k > 0)$ without the origin.



19. $k = \sqrt{x + y}$ or for $x + y \geq 0$, $k^2 = x + y$, or $y = -x + k^2$.
Note: $k \geq 0$ since $k = \sqrt{x + y}$.



20. $k = y \sec x$ or $y = k \cos x$, $x \neq \frac{\pi}{2} + n\pi$ (n an integer)



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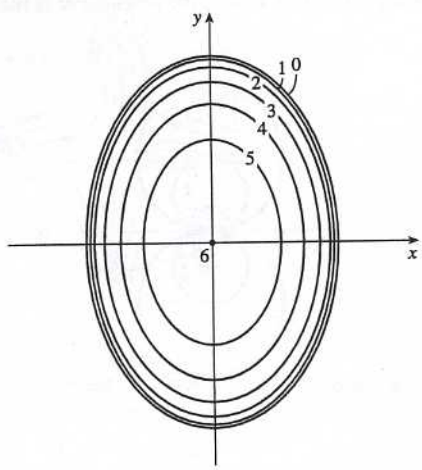
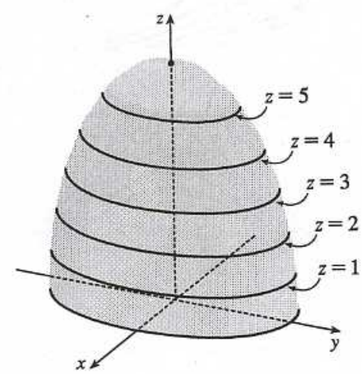
24. The contour map consists of the level curves $k = \sqrt{36 - 9x^2 - 4y^2} \;\Rightarrow\; 9x^2 + 4y^2 = 36 - k^2$, $k \geq 0$, a family of ellipses with major axis the y-axis. (Or, if $k = 6$, the origin.)

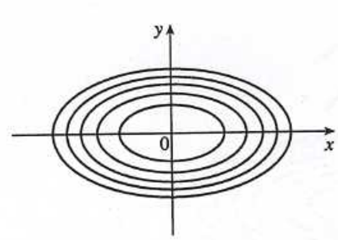
The graph of $f(x, y)$ is the surface $z = \sqrt{36 - 9x^2 - 4y^2}$, or equivalently the upper half of the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.





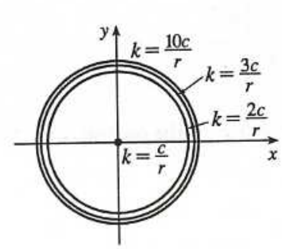
If we visualize lifting each ellipse $k = \sqrt{36 - 9x^2 - 4y^2}$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

25. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or $x^2 + 2y^2 = (100 - k)/k$ $(0 < k \leq 100)$, a family of ellipses.

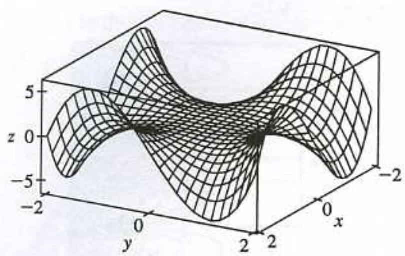


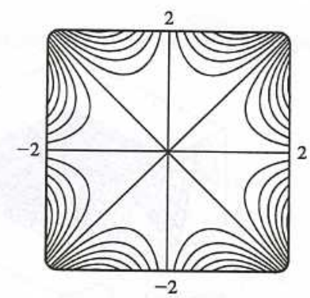
26. The equipotential curves are $k = c \Big/ \sqrt{r^2 - x^2 - y^2}$ or $x^2 + y^2 = r^2 - (c/k)^2$, a family of circles $(k \geq c/r)$.

Note: As $k \to \infty$, the radius of the circle approaches r.



30. $f(x,y) = xy^3 - yx^3$





The traces parallel to either the yz-plane or the xz-plane are cubic curves.

31. (a) B *Reasons:* This function is constant on any circle centered at the origin, a description which matches only B and III.
(b) III

32. (a) C *Reasons:* This function is the same if x is interchanged with y, so its graph is symmetric about the plane $x = y$. Also, $z(0,0) = 0$ and the values of z approach 0 as we use points farther from the origin. These conditions are satisfied only by C and II.
(b) II

33. (a) F *Reasons:* z increases without bound as we use points closer to the origin, a condition satisfied only by F and V.
(b) V

34. (a) A *Reasons:* Along the lines $y = \pm\frac{1}{\sqrt{3}}x$ and $x = 0$, this function is 0.
(b) VI

35. (a) D *Reasons:* This function is periodic in both x and y, with period 2π in each variable.
(b) IV

36. (a) E *Reasons:* This function is periodic along the x-axis, and increases as $|y|$ increases.
(b) I

37. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5 \rangle$.

38. $k = x^2 + 3y^2 + 5z^2$ is a family of ellipsoids for $k > 0$ and the origin for $k = 0$.

39. $k = x^2 - y^2 + z^2$ are the equations of the level surfaces. For $k = 0$, the surface is a right circular cone with vertex the origin and axis the y-axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the y-axis. For $k < 0$, we have a family of hyperboloids of two sheets with axis the y-axis.

40. $k = x^2 - y^2$ is a family of hyperbolic cylinders. The cross section of this family in the xy-plane has the same graph as the level curves in Exercise 16.

41. (a) The graph of g is the graph of f shifted upward 2 units.

(b) The graph of g is the graph of f stretched vertically by a factor of 2.

(c) The graph of g is the graph of f reflected about the xy-plane.

(d) The graph of $g(x,y) = -f(x,y) + 2$ is the graph of f reflected about the xy-plane and then shifted upward 2 units.

7. $f(x,y) = x^2/(x^2+y^2)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = x^2/x^2 = 1$ for $x \neq 0$, so $f(x,y) \to 1$. Now approach $(0,0)$ along the y-axis. Then for $y \neq 0$, $f(0,y) = 0$, so $f(x,y) \to 0$. Since f has two different limits along two different lines, the limit does not exist.

8. $f(x,y) = (x+y)^2/(x^2+y^2)$. As $(x,y) \to (0,0)$ along the x-axis, $f(x,y) \to 1$. But as $(x,y) \to (0,0)$ along the line $y = x$, $f(x,x) = 4x^2/(2x^2) = 2$ for $x \neq 0$, so $f(x,y) \to 2$. Thus, the limit does not exist.

9. $f(x,y) = 8x^2y^2/(x^4+y^4)$. Approaching $(0,0)$ along the x-axis gives $f(x,y) \to 0$. Approaching $(0,0)$ along the line $y = x$, $f(x,x) = 8x^4/2x^4 = 4$ for $x \neq 0$, so along this line $f(x,y) \to 4$ as $(x,y) \to (0,0)$. Thus the limit doesn't exist.

10. $\lim\limits_{(x,y)\to(0,0)} (x^3+xy^2)/(x^2+y^2) = \lim\limits_{(x,y)\to(0,0)} x = 0$

11. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. $0 \leq \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \leq |x|$ since $|y| \leq \sqrt{x^2+y^2}$, and $|x| \to 0$ as $(x,y) \to (0,0)$. So $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

12. We can use the Squeeze Theorem to show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 \sin^2 y}{x^2+2y^2} = 0$:

$0 \leq \dfrac{x^2\sin^2 y}{x^2+2y^2} \leq \sin^2 y$ since $\dfrac{x^2}{x^2+2y^2} \leq 1$, and $\sin^2 y \to 0$ as $(x,y) \to (0,0)$, so $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2+2y^2} = 0$.

13. Let $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x^2) = \dfrac{2x^4}{2x^4} = 1$ for $x \neq 0$, so $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$. Thus the limit doesn't exist.

14. $f(x,y) = \dfrac{xy-2y}{x^2+y^2-4x+4} = \dfrac{y(x-2)}{y^2+(x-2)^2}$. Then $f(x,0) = 0$ for $x \neq 2$, so $f(x,y) \to 0$ as $(x,y) \to (2,0)$ along the x-axis. But $f(x,x-2) = \dfrac{(x-2)(x-2)}{(x-2)^2+(x-2)^2} = \dfrac{(x-2)^2}{2(x-2)^2} = \dfrac{1}{2}$ for $x \neq 2$, so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (2,0)$ along the line $y = x-2$ $(x \neq 2)$. Thus, the limit doesn't exist.

15.
$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}\\ &= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2}\\ &= \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right) = 2\end{aligned}$$

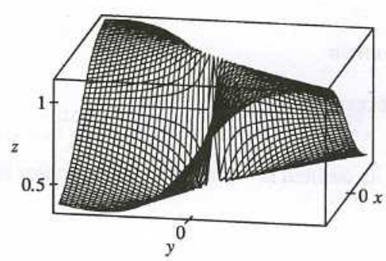
16. $e^{x^2 z}\cos(y+z)$ is a continuous function, so

$$\lim_{(x,y,z)\to(3,-2,2)} e^{x^2 z}\cos(y+z) = e^{3^2\cdot 2}\cos(-2+2) = e^{18}\cos 0 = e^{18}.$$

17. $f(x,y,z) = \dfrac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$. Then $f(x,0,0) = 0/x^2 = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

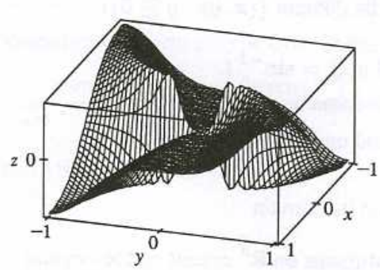
18. $f(x,y,z) = \dfrac{x^2 + 2y^2 + 3z^2}{x^2 + y^2 + z^2}$. Then $f(x,0,0) = \dfrac{x^2 + 0 + 0}{x^2 + 0 + 0} = 1$ for $x \neq 0$, so $f(x,y,z) \to 1$ as $(x,y,z) \to (0,0,0)$ along the x-axis. But $f(0,y,0) = \dfrac{0 + 2y^2 + 0}{0 + y^2 + 0} = 2$ for $y \neq 0$, so $f(x,y,z) \to 2$ as $(x,y,z) \to (0,0,0)$ along the y-axis. Thus, the limit doesn't exist.

19.



From the ridges on the graph, we see that as $(x,y) \to (0,0)$ along the lines under the two ridges, $f(x,y)$ approaches different values. So the limit does not exist.

20.

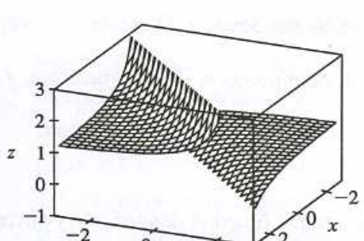


From the graph, it appears that as we approach the origin along the lines $x = 0$ or $y = 0$, the function is everywhere 0, whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. [In fact, $f(y^3, y) = y^6/(2y^6) = \frac{1}{2}$ for $y \neq 0$, so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along the curve $x = y^3$.] Since the function approaches different values depending on the path of approach, the limit does not exist.

21. $h(x,y) = g(f(x,y)) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus h is continuous on its domain $D = \{(x,y) \mid 2x + 3y - 6 \geq 0\} = \{(x,y) \mid y \geq -\frac{2}{3}x + 2\}$, which consists of all points on or above the line $y = -\frac{2}{3}x + 2$.

22. $h(x,y) = g(f(x,y)) = \sin(y \ln x)$. Since $f(x,y) = y \ln x$ it is continuous on its domain $\{(x,y) \mid x > 0\}$ and g is continuous throughout $\mathbb{R}$. Thus h is continuous on its domain $D = \{(x,y) \mid x > 0\}$, the right half-plane excluding the y-axis.

23.



From the graph, it appears that f is discontinuous along the line $y = x$. If we consider $f(x,y) = e^{1/(x-y)}$ as a composition of functions, $g(x,y) = 1/(x - y)$ is a rational function and therefore continuous except where $x - y = 0 \Rightarrow y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x,y)) = e^{1/(x-y)} = f(x,y)$ is continuous except along the line $y = x$, as we suspected.

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$f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

33. $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^3 + (r\sin\theta)^3}{r^2} = \lim_{r\to 0^+} \left(r\cos^3\theta + r\sin^3\theta\right) = 0$$

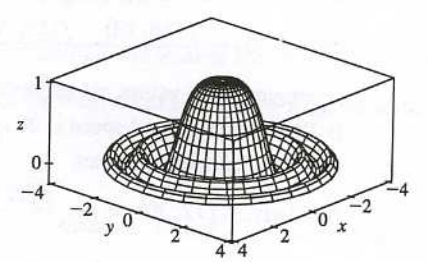
34. $$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0^+} r^2\ln r^2 = \lim_{r\to 0^+} \frac{\ln r^2}{1/r^2}$$
$$= \lim_{r\to 0^+} \frac{(1/r^2)(2r)}{-2/r^3} \text{ (using l'Hospital's Rule)} = \lim_{r\to 0^+} (-r^2) = 0$$

35. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0^+} \frac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$$
$$= \lim_{\rho\to 0^+} \left(\rho\sin^2\phi\cos\phi\sin\theta\cos\theta\right) = 0$$

36. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r\to 0^+} \dfrac{\sin(r^2)}{r^2}$, which is an indeterminate form of type $0/0$. Using l'Hospital's Rule, we get

$$\lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \overset{\text{H}}{=} \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = 1.$$

Or: Use the fact that $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.



11.3 Partial Derivatives

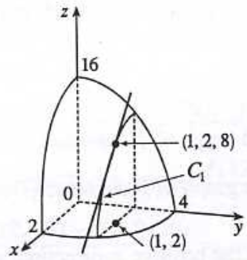
1. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

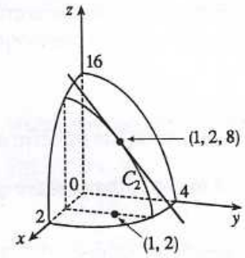
(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158° W, latitude 21° N at 9:00 A.M. when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as

8. $f_x(2,1)$ is the rate of change of f at $(2,1)$ in the x-direction. If we start at $(2,1)$, where $f(2,1) = 10$, and move in the positive x-direction, we reach the next contour line (where $f(x,y) = 12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive x-direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1) \approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of f at $(2,1)$ in the y-direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2. If we start at $(2,1)$ and move in the positive y-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1) \approx -2.1$.

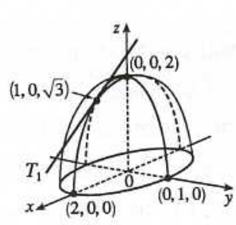
9. $f(x,y) = 16 - 4x^2 - y^2 \quad \Rightarrow \quad f_x(x,y) = -8x$ and $f_y(x,y) = -2y \quad \Rightarrow \quad f_x(1,2) = -8$ and $f_y(1,2) = -4$. The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$ (the curve C_1 in the first figure).

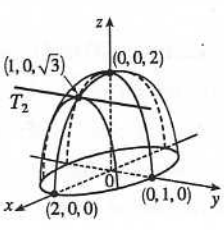
The slope of the tangent line to this parabola at $(1,2,8)$ is $f_x(1,2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1,2,8)$ is $f_y(1,2) = -4$.





10. $f(x,y) = \left(4 - x^2 - 4y^2\right)^{1/2} \quad \Rightarrow \quad f_x(x,y) = -x\left(4 - x^2 - 4y^2\right)^{-1/2}$ and $f_y(x,y) = -4y\left(4 - x^2 - 4y^2\right)^{-1/2} \quad \Rightarrow \quad f_x(1,0) = -\frac{1}{\sqrt{3}}$, $f_y(1,0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $\left(1, 0, \sqrt{3}\right)$ is $f_x(1,0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $\left(1, 0, \sqrt{3}\right)$ is $f_y(1,0) = 0$.





15. $z = xe^{3y} \;\Rightarrow\; \dfrac{\partial z}{\partial x} = e^{3y},\ \dfrac{\partial z}{\partial y} = 3xe^{3y}$

16. $z = y \ln x \;\Rightarrow\; \dfrac{\partial z}{\partial x} = \dfrac{y}{x},\ \dfrac{\partial z}{\partial y} = \ln x$

17. $f(x,y) = \dfrac{x-y}{x+y} \;\Rightarrow\; f_x(x,y) = \dfrac{(1)(x+y)-(x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2},$

$f_y(x,y) = \dfrac{(-1)(x+y)-(x-y)(1)}{(x+y)^2} = -\dfrac{2x}{(x+y)^2}$

18. $f(x,y) = x^y \;\Rightarrow\; f_x(x,y) = yx^{y-1},\ f_y(x,y) = x^y \ln x$

19. $w = \sin\alpha \cos\beta \;\Rightarrow\; \dfrac{\partial w}{\partial \alpha} = \cos\alpha\cos\beta,\ \dfrac{\partial w}{\partial \beta} = -\sin\alpha\sin\beta$

20. $f(s,t) = \dfrac{st^2}{s^2+t^2} \;\Rightarrow\; f_s(s,t) = \dfrac{t^2(s^2+t^2) - st^2(2s)}{(s^2+t^2)^2} = \dfrac{t^4 - s^2t^2}{(s^2+t^2)^2},$

$f_t(s,t) = \dfrac{2st(s^2+t^2) - st^2(2t)}{(s^2+t^2)^2} = \dfrac{2s^3t}{(s^2+t^2)^2}$

21. $f(u,v) = \tan^{-1}\left(\dfrac{u}{v}\right) \;\Rightarrow\; f_u(u,v) = \dfrac{1}{1+(u/v)^2}\left(\dfrac{1}{v}\right) = \dfrac{1}{v}\left(\dfrac{v^2}{u^2+v^2}\right) = \dfrac{v}{u^2+v^2},$

$f_v(u,v) = \dfrac{1}{1+(u/v)^2}\left(-\dfrac{u}{v^2}\right) = -\dfrac{u}{v^2}\left(\dfrac{v^2}{u^2+v^2}\right) = -\dfrac{u}{u^2+v^2}$

22. $f(x,t) = e^{\sin(t/x)} \;\Rightarrow\; f_x(x,t) = e^{\sin(t/x)}\cos\left(\dfrac{t}{x}\right)\left(-\dfrac{t}{x^2}\right) = -t\cos\left(\dfrac{t}{x}\right)\dfrac{e^{\sin(t/x)}}{x^2},$

$f_t(x,t) = e^{\sin(t/x)}\cos\left(\dfrac{t}{x}\right)\left(\dfrac{1}{x}\right) = \dfrac{e^{\sin(t/x)}}{x}\cos\left(\dfrac{t}{x}\right)$

23. $z = \ln\left(x + \sqrt{x^2+y^2}\right) \;\Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{1}{x+\sqrt{x^2+y^2}}\left[1 + \tfrac{1}{2}(x^2+y^2)^{-1/2}(2x)\right] = \frac{\left(\sqrt{x^2+y^2}+x\right)\Big/\sqrt{x^2+y^2}}{\left(x+\sqrt{x^2+y^2}\right)} = \frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+\sqrt{x^2+y^2}}\left(\frac{1}{2}\right)(x^2+y^2)^{-1/2}(2y) = \frac{y}{x\sqrt{x^2+y^2}+x^2+y^2}$$

24. $f(x,y) = \displaystyle\int_y^x \cos(t^2)\,dt \;\Rightarrow\; f_x(x,y) = \dfrac{\partial}{\partial x}\int_y^x \cos(t^2)\,dt = \cos(x^2)$ by the Fundamental Theorem of Calculus, Part 1; $f_y(x,y) = \dfrac{\partial}{\partial y}\displaystyle\int_y^x \cos(t^2)\,dt = -\dfrac{\partial}{\partial y}\int_x^y \cos(t^2)\,dt = -\cos(y^2)$.

25. $f(x,y,z) = xy^2z^3 + 3yz \;\Rightarrow\; f_x(x,y,z) = y^2z^3,\ f_y(x,y,z) = 2xyz^3 + 3z,\ f_z(x,y,z) = 3xy^2z^2 + 3y$

26. $f(x,y,z) = x^2e^{yz} \;\Rightarrow\; f_x(x,y,z) = 2xe^{yz},\ f_y(x,y,z) = x^2e^{yz}(z) = x^2ze^{yz},$

$f_z(x,y,z) = x^2e^{yz}(y) = x^2ye^{yz}$.

28. $w = \sqrt{r^2+s^2+t^2} \;\Rightarrow\; \dfrac{\partial w}{\partial r} = \dfrac{1}{2}\left(r^2+s^2+t^2\right)^{-1/2}(2r) = \dfrac{r}{\sqrt{r^2+s^2+t^2}}$, $\dfrac{\partial w}{\partial s} = \dfrac{s}{\sqrt{r^2+s^2+t^2}}$,

$\dfrac{\partial w}{\partial t} = \dfrac{t}{\sqrt{r^2+s^2+t^2}}$.

29. $u = xe^{-t}\sin\theta \;\Rightarrow\; \dfrac{\partial u}{\partial x} = e^{-t}\sin\theta$, $\dfrac{\partial u}{\partial t} = -xe^{-t}\sin\theta$, $\dfrac{\partial u}{\partial \theta} = xe^{-t}\cos\theta$

30. $u = x^{y/z} \;\Rightarrow\; u_x = \dfrac{y}{z}x^{(y/z)-1}$, $u_y = x^{y/z}\ln x\cdot\dfrac{1}{z} = \dfrac{x^{y/z}}{z}\ln x$, $u_z = x^{y/z}\ln x\cdot\dfrac{-y}{z^2} = -\dfrac{yx^{y/z}}{z^2}\ln x$

31. $f(x,y,z,t) = \dfrac{x-y}{z-t} \;\Rightarrow\; f_x(x,y,z,t) = \dfrac{1}{z-t}$, $f_y(x,y,z,t) = -\dfrac{1}{z-t}$,

$f_z(x,y,z,t) = (x-y)(-1)(z-t)^{-2} = \dfrac{y-x}{(z-t)^2}$, and

$f_t(x,y,z,t) = (x-y)(-1)(z-t)^{-2}(-1) = \dfrac{x-y}{(z-t)^2}$.

32. $f(x,y,z,t) = xy^2z^3t^4 \;\Rightarrow\; f_x(x,y,z,t) = y^2z^3t^4$, $f_y(x,y,z,t) = 2xyz^3t^4$, $f_z(x,y,z,t) = 3xy^2z^2t^4$, and $f_t(x,y,z,t) = 4xy^2z^3t^3$.

33. $u = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$. For each $i = 1,\ldots,n$,

$u_{x_i} = \frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$.

34. $u = \sin(x_1+2x_2+\cdots+nx_n)$. For each $i = 1,\ldots,n$, $u_{x_i} = i\cos(x_1+2x_2+\cdots+nx_n)$.

35. $f(x,y) = \sqrt{x^2+y^2} \;\Rightarrow\; f_x(x,y) = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+y^2}}$, so $f_x(3,4) = \dfrac{3}{\sqrt{3^2+4^2}} = \dfrac{3}{5}$.

36. $f(x,y) = \sin(2x+3y) \;\Rightarrow\; f_y(x,y) = \cos(2x+3y)\cdot 3 = 3\cos(2x+3y)$, so $f_y(-6,4) = 3\cos[2(-6)+3(4)] = 3\cos 0 = 3$.

37. $f(x,y,z) = \dfrac{x}{y+z} = x(y+z)^{-1} \;\Rightarrow\; f_z(x,y,z) = x(-1)(y+z)^{-2} = -\dfrac{x}{(y+z)^2}$, so

$f_z(3,2,1) = -\dfrac{3}{(2+1)^2} = -\dfrac{1}{3}$.

38. $f(u,v,w) = w\tan(uv) \;\Rightarrow\; f_v(u,v,w) = w\sec^2(uv)\cdot u = uw\sec^2(uv)$, so $f_v(2,0,3) = (2)(3)\sec^2(2\cdot 0) = 6$.

39. $f(x,y) = x^2 - xy + 2y^2 \;\Rightarrow$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x+h)y+2y^2-\left(x^2-xy+2y^2\right)}{h}$$

$$= \lim_{h\to 0}\frac{h(2x-y+h)}{h} = \lim_{h\to 0}(2x-y+h) = 2x-y$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0}\frac{x^2-x(y+h)+2(y+h)^2-\left(x^2-xy+2y^2\right)}{h}$$

$$= \lim_{h\to 0}\frac{h(4y-x+2h)}{h} = \lim_{h\to 0}(4y-x+2h) = 4y-x$$

45. (a) $z = f(x) + g(y) \;\Rightarrow\; \dfrac{\partial z}{\partial x} = f'(x),\ \dfrac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x + y)$. Let $u = x + y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}(1) = f'(u) = f'(x + y)$,
$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du}(1) = f'(u) = f'(x + y)$.

46. (a) $z = f(x)g(y) \;\Rightarrow\; \dfrac{\partial z}{\partial x} = f'(x)g(y),\ \dfrac{\partial z}{\partial y} = f(x)g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\dfrac{\partial u}{\partial x} = y$ and $\dfrac{\partial u}{\partial y} = x$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du} \cdot y = yf'(u) = yf'(xy)$ and
$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du} \cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\dfrac{x}{y}\right)$. Let $u = \dfrac{x}{y}$. Then $\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = f'(u)\dfrac{1}{y} = \dfrac{f'(x/y)}{y}$ and
$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = f'(u)\left(-\dfrac{x}{y^2}\right) = -\dfrac{xf'(x/y)}{y^2}$.

47. $f(x, y) = x^4 - 3x^2y^3 \;\Rightarrow\; f_x(x, y) = 4x^3 - 6xy^3,\ f_y(x, y) = -9x^2y^2$. Then $f_{xx}(x, y) = 12x^2 - 6y^3$,
$f_{xy}(x, y) = -18xy^2,\ f_{yx}(x, y) = -18xy^2$, and $f_{yy}(x, y) = -18x^2y$.

48. $f(x, y) = \ln(3x + 5y) \;\Rightarrow\; f_x(x, y) = \dfrac{3}{3x + 5y},\ f_y(x, y) = \dfrac{5}{3x + 5y}$. Then
$f_{xx}(x, y) = 3(-1)(3x + 5y)^{-2}(3) = -\dfrac{9}{(3x + 5y)^2},\ f_{xy}(x, y) = -\dfrac{15}{(3x + 5y)^2},\ f_{yx}(x, y) = -\dfrac{15}{(3x + 5y)^2}$,
and $f_{yy}(x, y) = -\dfrac{25}{(3x + 5y)^2}$.

49. $u = e^{-s}\sin t \;\Rightarrow\; u_s = -e^{-s}\sin t,\ u_t = e^{-s}\cos t$. Then $u_{ss} = e^{-s}\sin t,\ u_{st} = -e^{-s}\cos t$,
$u_{ts} = -e^{-s}\cos t$, and $u_{tt} = -e^{-s}\sin t$.

50. $z = y\tan 2x \;\Rightarrow\; z_x = y\sec^2(2x) \cdot 2 = 2y\sec^2(2x),\ z_y = \tan 2x$. Then
$z_{xx} = 2y(2)\sec(2x) \cdot \sec(2x)\tan(2x) \cdot 2 = 8y\sec^2(2x)\tan(2x),\ z_{xy} = 2\sec^2(2x)$,
$z_{yx} = \sec^2(2x) \cdot 2 = 2\sec^2(2x)$, and $z_{yy} = 0$.

51. $u = \ln\sqrt{x^2 + y^2} = \ln(x^2 + y^2)^{1/2} = \frac{1}{2}\ln(x^2 + y^2) \;\Rightarrow\; u_x = \dfrac{1}{2}\dfrac{1}{x^2 + y^2} \cdot 2x = \dfrac{x}{x^2 + y^2}$,
$u_{xy} = x(-1)(x^2 + y^2)^{-2}(2y) = -\dfrac{2xy}{(x^2 + y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2 + y^2} \cdot 2y = \dfrac{y}{x^2 + y^2}$,
$u_{yx} = y(-1)(x^2 + y^2)^{-2}(2x) = -\dfrac{2xy}{(x^2 + y^2)^2}$. Thus $u_{xy} = u_{yx}$.

52. $u = xye^y \;\Rightarrow\; u_x = ye^y,\ u_{xy} = ye^y + e^y = (y + 1)e^y$ and $u_y = x(ye^y + e^y) = x(y + 1)e^y$,
$u_{yx} = (y + 1)e^y$. Thus $u_{xy} = u_{yx}$.

53. $f(x, y) = x^2y^3 - 2x^4y \;\Rightarrow\; f_x = 2xy^3 - 8x^3y,\ f_{xx} = 2y^3 - 24x^2y,\ f_{xxx} = -48xy$

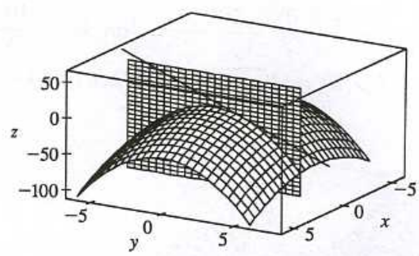
54. $f(x, y) = e^{xy^2} \;\Rightarrow\; f_x = y^2e^{xy^2},\ f_{xx} = y^4e^{xy^2},\ f_{xxy} = 4y^3e^{xy^2} + 2xy^5e^{xy^2} = 2y^3e^{xy^2}(2 + xy^2)$

55. $f(x, y, z) = x^5 + x^4y^4z^3 + yz^2 \;\Rightarrow\; f_x = 5x^4 + 4x^3y^4z^3,\ f_{xy} = 16x^3y^3z^3$, and $f_{xyz} = 48x^3y^3z^2$

56. $f(x, y, z) = e^{xyz} \;\Rightarrow\; f_y = xze^{xyz},\ f_{yz} = xe^{xyz} + xz(xy)e^{xyz} = xe^{xyz}(1 + yxz)$, and
$f_{yzy} = x(xz)e^{xyz}(1 + yxz) + xe^{xyz}(xz) = x^2z(2 + xyz)e^{xyz}$.

73. $f_x(x,y) = x + 4y \;\Rightarrow\; f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \;\Rightarrow\; f_{yx}(x,y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

74. Setting $x = 1$, the equation of the parabola of intersection is $z = 6 - 1 - 1 - 2y^2 = 4 - 2y^2$. The slope of the tangent is $\partial z/\partial y = -4y$, so at $(1,2,-4)$ the slope is -8. Parametric equations for the line are therefore $x = 1$, $y = 2 + t$, $z = -4 - 8t$.



75. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \;\Rightarrow\; \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x - 1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

76. $T(x,t) = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$

(a)
$$\begin{aligned}\partial T/\partial x &= T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x)\,(-\lambda)\right] + T_1\left(-\lambda e^{-\lambda x}\right)\sin(\omega t - \lambda x)\\ &= -\lambda T_1 e^{-\lambda x}\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]\end{aligned}$$

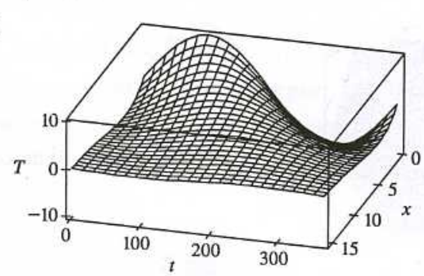
This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t.

(b) $\partial T/\partial t = T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x)\,(\omega)\right] = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x.

(c)
$$\begin{aligned}T_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\\ &= -\lambda T_1\big(e^{-\lambda x}\left[\cos(\omega t - \lambda x)\,(-\lambda) - \sin(\omega t - \lambda x)\,(-\lambda)\right]\\ &\qquad\qquad + e^{-\lambda x}\,(-\lambda)\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]\big)\\ &= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t - \lambda x)\end{aligned}$$

But from part (b), $T_t = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x) = \dfrac{\omega}{2\lambda^2} T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t \approx 100$, whereas at a depth of 5 feet the peak temperature is attained at $t \approx 150$, and at a depth of 10 feet, at $t \approx 220$.