11.4 Tangent Planes and Linear Approximations

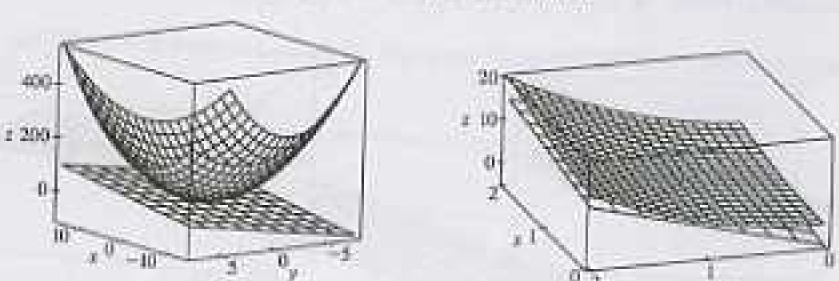
1. $z = f(x,y) = 4x^2 - y^2 + 2y \;\Rightarrow\; f_x(x,y) = 8x$, $f_y(x,y) = -2y + 2$, so $f_x(-1,2) = -8$, $f_y(-1,2) = -2$. By Equation 2, an equation of the tangent plane is $z - 4 = f_x(-1,2)[x - (-1)] + f_y(-1,2)(y - 2) \;\Rightarrow$ $z - 4 = -8(x+1) - 2(y-2)$ or $z = -8x - 2y$.

2. $z = f(x,y) = e^{x^2-y^2} \;\Rightarrow\; f_x(x,y) = 2xe^{x^2-y^2}$, $f_y(x,y) = -2ye^{x^2-y^2}$, so $f_x(1,-1) = 2$, $f_y(1,-1) = 2$. By Equation 2, an equation of the tangent plane is $z - 1 = f_x(1,-1)(x-1) + f_y(1,-1)[y - (-1)] \;\Rightarrow$ $z - 1 = 2(x-1) + 2(y+1)$ or $z = 2x + 2y + 1$.

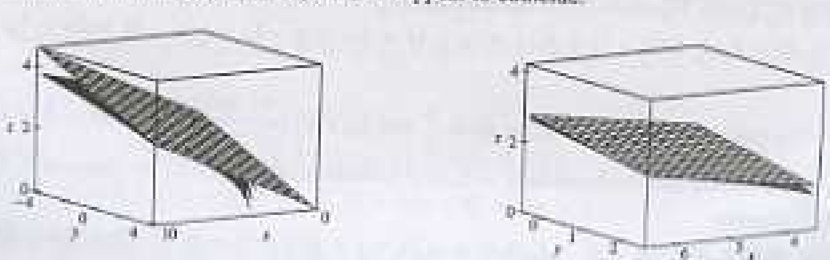
3. $z = f(x,y) = \sqrt{4 - x^2 - 2y^2} \;\Rightarrow\; f_x(x,y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-2x) = -\dfrac{x}{\sqrt{4 - x^2 - 2y^2}}$, $f_y(x,y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-4y) = -\dfrac{2y}{\sqrt{4 - x^2 - 2y^2}}$, so $f_x(1,-1) = -1$ and $f_y(1,-1) = 2$. Thus, an equation of the tangent plane is $z - 1 = f_x(1,-1)(x-1) + f_y(1,-1)[y - (-1)] \;\Rightarrow$ $z - 1 = -1(x-1) + 2(y+1)$ or $x - 2y + z = 4$.

4. $z = f(x,y) = y \ln x \;\Rightarrow\; f_x(x,y) = y/x$, $f_y(x,y) = \ln x$, so $f_x(1,4) = 4$, $f_y(1,4) = 0$, and an equation of the tangent plane is $z - 0 = f_x(1,4)(x-1) + f_y(1,4)(y-4) \;\Rightarrow\; z = 4(x-1) + 0(y-4)$ or $z = 4x - 4$.

5. $z = f(x,y) = x^2 + xy + 3y^2$, so $f_x(x,y) = 2x + y \;\Rightarrow\; f_x(1,1) = 3$, $f_y(x,y) = x + 6y \;\Rightarrow\; f_y(1,1) = 7$ and an equation of the tangent plane is $z - 5 = 3(x-1) + 7(y-1)$ or $z = 3x + 7y - 5$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



6. $z = f(x,y) = \sqrt{x - y}$, so $f_x(x,y) = \frac{1}{2}(x-y)^{-1/2}$, $f_x(5,1) = \frac{1}{4}$, $f_y(x,y) = -\frac{1}{2}(x-y)^{-1/2}$, $f_y(5,1) = -\frac{1}{4}$, and an equation of the tangent plane is $z - 2 = \frac{1}{4}(x-5) - \frac{1}{4}(y-1)$ or $z = \frac{1}{4}x - \frac{1}{4}y + 1$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



11. $f(x,y)=\tan^{-1}(x+2y)$. The partial derivatives are $f_x(x,y)=\dfrac{1}{1+(x+2y)^2}$ and $f_y(x,y)=\dfrac{2}{1+(x+2y)^2}$, so $f_x(1,0)=\frac{1}{2}$ and $f_y(1,0)=1$. Both f_x and f_y are continuous functions, so f is differentiable at $(1,0)$, and the linearization of f at $(1,0)$ is $L(x,y)=f(1,0)+f_x(1,0)(x-1)+f_y(1,0)(y-0)=\frac{\pi}{4}+\frac{1}{2}(x-1)+1(y)=\frac{1}{2}x+y+\frac{\pi}{4}-\frac{1}{2}$.

12. $f(x,y)=\sin(2x+3y)$. The partial derivatives are $f_x(x,y)=2\cos(2x+3y)$ and $f_y(x,y)=3\cos(2x+3y)$, so $f_x(-3,2)=2$ and $f_y(-3,2)=3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3,2)$, and the linearization of f at $(-3,2)$ is $L(x,y)=f(-3,2)+f_x(-3,2)(x+3)+f_y(-3,2)(y-2)=0+2(x+3)+3(y-2)=2x+3y$.

13. $f(x,y)=\sqrt{20-x^2-7y^2}\ \Rightarrow\ f_x(x,y)=-\dfrac{x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y)=-\dfrac{7y}{\sqrt{20-x^2-7y^2}}$, so $f_x(2,1)=-\frac{2}{3}$ and $f_y(2,1)=-\frac{7}{3}$. Then the linear approximation of f at $(2,1)$ is given by

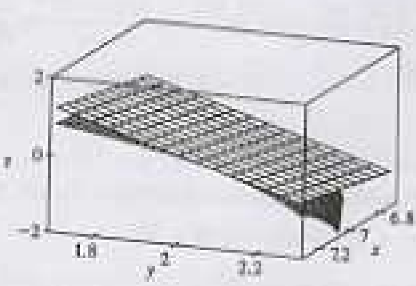
$$\begin{aligned} f(x,y)&\approx f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1)=3-\tfrac{2}{3}(x-2)-\tfrac{7}{3}(y-1)\\ &=-\tfrac{2}{3}x-\tfrac{7}{3}y+\tfrac{20}{3}\end{aligned}$$

Thus $f(1.95,1.08)\approx-\frac{2}{3}(1.95)-\frac{7}{3}(1.08)+\frac{20}{3}=2.846$.

14. $f(x,y)=\ln(x-3y)\ \Rightarrow\ f_x(x,y)=\dfrac{1}{x-3y}$ and $f_y(x,y)=-\dfrac{3}{x-3y}$, so $f_x(7,2)=1$ and $f_y(7,2)=-3$. Then the linear approximation of f at $(7,2)$ is given by

$$\begin{aligned} f(x,y)&\approx f(7,2)+f_x(7,2)(x-7)+f_y(7,2)(y-2)\\ &=0+1(x-7)-3(y-2)=x-3y-1\end{aligned}$$

Thus $f(6.9,2.06)\approx 6.9-3(2.06)-1=-0.28$. The graph shows that our approximated value is slightly greater than the actual value.



15. $f(x,y,z)=\sqrt{x^2+y^2+z^2}\ \Rightarrow\ f_x(x,y,z)=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x,y,z)=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and $f_z(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6)=\frac{3}{7}$, $f_y(3,2,6)=\frac{2}{7}$, and $f_z(3,2,6)=\frac{6}{7}$. Then the linear approximation of f at $(3,2,6)$ is given by

$$\begin{aligned} f(x,y,z)&\approx f(3,2,6)+f_x(3,2,6)(x-3)+f_y(3,2,6)(y-2)+f_z(3,2,6)(z-6)\\ &=7+\tfrac{3}{7}(x-3)+\tfrac{2}{7}(y-2)+\tfrac{6}{7}(z-6)=\tfrac{3}{7}x+\tfrac{2}{7}y+\tfrac{6}{7}z\end{aligned}$$

Thus $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2}=f(3.02,1.97,5.99)\approx\frac{3}{7}(3.02)+\frac{2}{7}(1.97)+\frac{6}{7}(5.99)\approx 6.9914$.

Averaging these values gives $f_T(16, 30) \approx 1.375$. Similarly, $f_v(16, 30) = \lim\limits_{h \to 0} \dfrac{f(16, 30) - f(16, 30)}{h}$, so we use $h = \pm 10$: $f_v(16, 30) \approx \dfrac{f(16, 40) - f(16, 30)}{10} = \dfrac{7 - 9}{10} = -0.2$, $f_v(16, 30) \approx \dfrac{f(16, 20) - f(16, 30)}{-10} = \dfrac{11 - 9}{-10} = -0.2$. Averaging these values gives $f_v(16, 30) \approx -0.2$. The linear approximation, then, is

$$\begin{aligned} f(T, v) &\approx f(16, 30) + f_T(16, 30)(T - 16) + f_v(16, 30)(v - 30) \\ &\approx 9 + 1.375(T - 16) - 0.2(v - 30) \end{aligned}$$

Thus when $T = 14$ and $v = 27$, $f(14, 27) \approx 9 + 1.375(14 - 16) - 0.2(27 - 30) = 6.85$, so we estimate the wind-chill index to be approximately $6.85\,^\circ$C.

19. $u = e^t \sin\theta \;\Rightarrow\; du = \dfrac{\partial u}{\partial t}\,dt + \dfrac{\partial u}{\partial \theta}\,d\theta = e^t \sin\theta\,dt + e^t \cos\theta\,d\theta$

20. $v = y\cos xy \;\Rightarrow$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = y(-\sin xy)y\,dx + [y(-\sin xy)x + \cos xy]\,dy = -y^2 \sin xy\,dx + (\cos xy - xy\sin xy)\,dy$$

21. $w = \ln\sqrt{x^2 + y^2 + z^2} \;\Rightarrow$

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz \\ &= \left(\frac{1}{2}\right)\frac{2x\,(x^2 + y^2 + z^2)^{-1/2}\,dx + 2y(x^2 + y^2 + z^2)^{-1/2}\,dy + 2z(x^2 + y^2 + z^2)^{-1/2}\,dz}{(x^2 + y^2 + z^2)^{1/2}} \\ &= \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2} \end{aligned}$$

22. $u = \dfrac{r}{s + 2t} \;\Rightarrow$

$$\begin{aligned} du &= \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t}dt = \frac{1}{s + 2t}\,dr + r(-1)(s + 2t)^{-2}\,ds + r(-1)(s + 2t)^{-2}(2)\,dt \\ &= \frac{1}{s + 2t}\,dr - \frac{r}{(s + 2t)^2}\,ds - \frac{2r}{(s + 2t)^2}\,dt \end{aligned}$$

23. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$, $dz = z_x(1, 2)\,dx + z_y(1, 2)\,dy = (10)(0.05) + (4)(0.1) = 0.9$ while $\Delta z = f(1.05, 2.1) - f(1, 2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

24. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$, $dz = (7)(-0.04) + (-9)(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9 + 3 + 3) = -0.7189$.

25. $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$ and $|\Delta x| \le 0.1$, $|\Delta y| \le 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4\text{ cm}^2$.

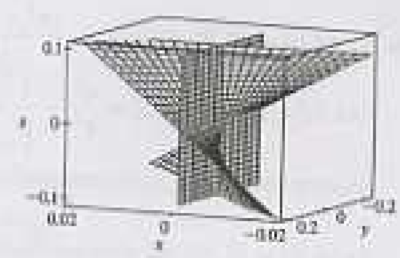
26. Let S be surface area. Then $S = 2(xy + xz + yz)$ and $dS = 2(y + z)\,dx + 2(x + z)\,dy + 2(x + y)\,dz$. The maximum error occurs with $\Delta x = \Delta y = \Delta z = 0.2$. Using $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$ we find the maximum error in calculated surface area to be about $dS = (220)(0.2) + (260)(0.2) + (280)(0.2) = 152\text{ cm}^2$.

35. $\mathbf{r}(u,v) = uv\,\mathbf{i} + ue^v\,\mathbf{j} + ve^u\,\mathbf{k}$

$\mathbf{r}_u = \langle v, e^v, ve^u\rangle$, $\mathbf{r}_v = \langle u, ue^v, e^u\rangle$, and

$\mathbf{r}_u \times \mathbf{r}_v = e^{u+v}(1-uv)\,\mathbf{i} + e^u(uv-v)\,\mathbf{j} + e^v(uv-u)\,\mathbf{k}$.

The point $(0,0,0)$ corresponds to $u=0$, $v=0$. Thus a normal vector to the surface at $(0,0,0)$ is $\mathbf{i}$ and an equation of the tangent plane is $x = 0$.

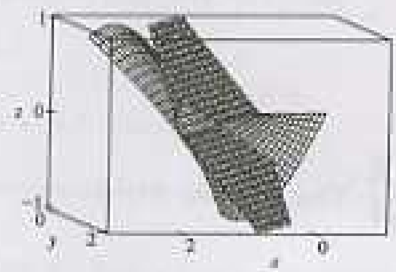


36. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + u\cos v\,\mathbf{j} + v\sin u\,\mathbf{k}$

$\mathbf{r}_u = \langle 1, \cos v, v\cos u\rangle$, $\mathbf{r}_v = \langle 1, -u\sin v, \sin u\rangle$, and

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v\sin u + uv\cos u\sin v,\; v\cos u - \sin u,\; -u\sin v - \cos v\rangle$$

The point $(1,1,0)$ corresponds to $u=1$, $v=0$. Thus a normal vector to the surface at $(1,1,0)$ is $\langle \sin 1, -\sin 1, -1\rangle$, and an equation of the tangent plane is $(\sin 1)x - (\sin 1)y - z = 0$.



37. $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) = (a+\Delta x)^2 + (b+\Delta y)^2 - (a^2+b^2)$

$= a^2 + 2a\,\Delta x + (\Delta x)^2 + b^2 + 2b\,\Delta y + (\Delta y)^2 - a^2 - b^2 = 2a\,\Delta x + (\Delta x)^2 + 2b\,\Delta y + (\Delta y)^2$

But $f_x(a,b) = 2a$ and $f_y(a,b) = 2b$ and so $\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \Delta x\,\Delta x + \Delta y\,\Delta y$, which is Definition 7 with $\epsilon_1 = \Delta x$ and $\epsilon_2 = \Delta y$. Hence f is differentiable.

38. $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) = (a+\Delta x)(b+\Delta y) - 5(b+\Delta y)^2 - (ab - 5b^2)$

$= ab + a\,\Delta y + b\,\Delta x + \Delta x\,\Delta y - 5b^2 - 10b\,\Delta y - 5\,(\Delta y)^2 - ab + 5b^2$

$= (a - 10b)\,\Delta y + b\,\Delta x + \Delta x\,\Delta y - 5\,\Delta y\,\Delta y$,

but $f_x(a,b) = b$ and $f_y(a,b) = a - 10b$ and so $\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \Delta x\,\Delta y - 5\Delta y\,\Delta y$, which is Definition 7 with $\epsilon_1 = \Delta y$ and $\epsilon_2 = -5\,\Delta y$. Hence f is differentiable.

39. To show that f is continuous at (a,b) we need to show that $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ or equivalently $\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$. Since f is differentiable at (a,b),

$f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$, where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Thus $f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \to (0,0)$ gives $\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$. Thus f is continuous at (a,b).

40. (a) $\lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$ and $\lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$. Thus $f_x(0,0) = f_y(0,0) = 0$. To show that f isn't differentiable at $(0,0)$ we need only show that f is not continuous at $(0,0)$ and apply Exercise 39. As $(x,y) \to (0,0)$ along the x-axis $f(x,y) = 0/x^2 = 0$ for $x \neq 0$ so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But as $(x,y) \to (0,0)$ along the line $y = x$, $f(x,x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along this line. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is discontinuous at $(0,0)$ and thus not differentiable there.

(b) For $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \dfrac{y(y^2-x^2)}{(x^2+y^2)^2}$. If we approach $(0,0)$ along the y-axis, then $f_x(x,y) = f_x(0,y) = \dfrac{y^3}{y^4} = \dfrac{1}{y}$, so $f_x(x,y) \to \pm\infty$ as $(x,y) \to (0,0)$. Thus $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ does not exist and $f_x(x,y)$ is not continuous at $(0,0)$. Similarly, $f_y(x,y) = \dfrac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \dfrac{x(x^2-y^2)}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$, and if we approach $(0,0)$ along the x-axis, then $f_y(x,y) = f_x(x,0) = \dfrac{x^3}{x^4} = \dfrac{1}{x}$. Thus $\lim\limits_{(x,y)\to(0,0)} f_y(x,y)$ does not exist and $f_y(x,y)$ is not continuous at $(0,0)$.

11.5 The Chain Rule

1. $z = \sin x \cos y$, $x = \pi t$, $y = \sqrt{t}$ $\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \cos x \cos y \cdot \pi + \sin x\,(-\sin y)\cdot \tfrac{1}{2}t^{-1/2} = \pi \cos x \cos y - \frac{1}{2\sqrt{t}}\sin x \sin y$$

2. $z = x\ln(x+2y)$, $x = \sin t$, $y = \cos t$ $\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left[x\cdot\frac{1}{x+2y} + 1\cdot\ln(x+2y)\right]\cos t + x\cdot\frac{1}{x+2y}(2)\cdot(-\sin t)$$
$$= \left[\frac{x}{x+2y} + \ln(x+2y)\right]\cos t - \frac{2x}{x+2y}(\sin t)$$

3. $w = xe^{y/z}$, $x = t^2$, $y = 1-t$, $z = 1+2t$ $\Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

4. $w = xy + yz^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$ $\Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = y\cdot e^t + (x+z^2)\cdot(e^t\cos t + e^t \sin t) + 2yz\cdot(-e^t\sin t + e^t\cos t)$$
$$= e^t\left[y + (x+z^2)(\cos t + \sin t) + 2yz(\cos t - \sin t)\right]$$

5. $z = x^2 + xy + y^2$, $x = s+t$, $y = st$ $\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x+y)(1) + (x+2y)(t) = 2x + y + xt + 2yt$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2x+y)(1) + (x+2y)(s) = 2x + y + xs + 2ys$$

6. $z = \dfrac{x}{y}$, $x = se^t$, $y = 1 + se^{-t}$ $\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{y}(e^t) + \left(-\frac{x}{y^2}\right)(e^{-t}) = \frac{1}{y}e^t - \frac{x}{y^2}e^{-t}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{y}(se^t) + \left(-\frac{x}{y^2}\right)(-se^{-t}) = \frac{s}{y}e^t + \frac{xs}{y^2}e^{-t}$$

7. $z = e^r \cos\theta,\ r = st,\ \theta = \sqrt{s^2+t^2} \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + e^r(-\sin\theta)\cdot \tfrac{1}{2}\left(s^2+t^2\right)^{-1/2}(2s)$$

$$= te^r\cos\theta - e^r\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}} = e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + e^r(-\sin\theta)\cdot \tfrac{1}{2}\left(s^2+t^2\right)^{-1/2}(2t)$$

$$= se^r\cos\theta - e^r\sin\theta\cdot\frac{t}{\sqrt{s^2+t^2}} = e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)$$

8. $z = \sin\alpha\tan\beta,\ \alpha = 3s+t,\ \beta = s-t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial s} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial s} = \cos\alpha\tan\beta\cdot 3 + \sin\alpha\sec^2\beta\cdot 1 = 3\cos\alpha\tan\beta + \sin\alpha\sec^2\beta$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial t} = \cos\alpha\tan\beta\cdot 1 + \sin\alpha\sec^2\beta\cdot(-1) = \cos\alpha\tan\beta - \sin\alpha\sec^2\beta$$

9. When $t = 3$, $x = g(3) = 2$ and $y = h(3) = 7$. By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(2,7)\,g'(3) + f_y(2,7)\,h'(3) = (6)(5) + (-8)(-4) = 62.$$

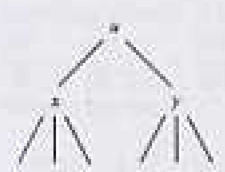
10. By the Chain Rule (3), $\dfrac{\partial W}{\partial s} = \dfrac{\partial W}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial W}{\partial v}\dfrac{\partial v}{\partial s}$. Then

$$W_s(1,0) = F_u\left(u(1,0), v(1,0)\right)u_s(1,0) + F_v\left(u(1,0), v(1,0)\right)v_s(1,0)$$
$$= F_u(2,3)u_s(1,0) + F_v(2,3)v_s(1,0) = (-1)(-2) + (10)(5) = 52$$

Similarly, $\dfrac{\partial W}{\partial t} = \dfrac{\partial W}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial W}{\partial v}\dfrac{\partial v}{\partial t} \quad \Rightarrow$

$$W_t(1,0) = F_u\left(u(1,0), v(1,0)\right)u_t(1,0) + F_v\left(u(1,0), v(1,0)\right)v_t(1,0)$$
$$= F_u(2,3)u_t(1,0) + F_v(2,3)v_t(1,0) = (-1)(6) + (10)(4) = 34$$

11.

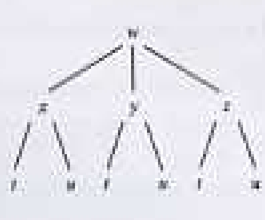


$u = f(x,y),\ x = x(r,s,t),\ y = y(r,s,t) \quad \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r},\quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

12.



$w = f(x,y,z),\ x = x(t,u),\ y = y(t,u),\ z = z(t,u) \quad \Rightarrow$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t},$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

21. $\cos(x-y) = xe^y$, so let $F(x,y) = \cos(x-y) - xe^y = 0$.
Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{-\sin(x-y) - e^y}{-\sin(x-y)(-1) - xe^y} = \dfrac{\sin(x-y) + e^y}{\sin(x-y) - xe^y}$.

22. $\sin x + \cos y = \sin x \cos y$, so let $F(x,y) = \sin x + \cos y - \sin x \cos y = 0$. Then
$\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{\cos x - \cos x \cos y}{-\sin y + \sin x \sin y} = \dfrac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}$.

23. $xy^2 + yz^2 + zx^2 = 3$, so let $F(x,y) = xy^2 + yz^2 + zx^2 - 3 = 0$.
Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{y^2 + 2zx}{2yz + x^2}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2xy + z^2}{2yz + x^2}$.

24. $xyz = \cos(x + y + z)$. Let $F(x,y,z) = xyz - \cos(x + y + z) = 0$, so
$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}$, $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$.

25. Let $F(x,y,z) = xe^y + yz + ze^x = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{e^y + ze^x}{y + e^x}$, $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{xe^y + z}{y + e^x}$.

26. $\ln(x + yz) = 1 + xy^2z^3$, so let $F(x,y) = \ln(x + yz) - 1 - xy^2z^3 = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1/(x+yz) - y^2z^3}{y/(x+yz) - 3xy^2z^2} = \frac{y^2z^3(x+yz) - 1}{y - 3xy^2z^2(x+yz)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z/(x+yz) - 2xyz^3}{y/(x+yz) - 3xy^2z^2} = \frac{2xyz^3(x+yz) - z}{y - 3xy^2z^2(x+yz)}$$

27. Since x and y are each functions of t, $T(x,y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$.
After 3 seconds, $x = \sqrt{1+t} = \sqrt{1+3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1+t}} = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$,
and $\dfrac{dy}{dt} = \dfrac{1}{3}$. Then $\dfrac{dT}{dt} = T_x(2,3)\dfrac{dx}{dt} + T_y(2,3)\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of 2 °C/s.

28. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of 0.15 °C/year, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year, we know $dR/dt = -0.1$. Then, by the Chain Rule,
$\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.

29. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and
$\dfrac{\partial C}{\partial D} = 0.016$. According to the graph, the diver is experiencing a temperature of approximately 12.5 °C at $t = 20$ minutes, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate $\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule,
$\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$. Thus the speed of sound experienced by the diver is changing at a rate of approximately 0.33 m/s per minute.

30. $V = \pi r^2 h/3$, so $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = \dfrac{2\pi r h}{3}\,1.8 + \dfrac{\pi r^2}{3}(-2.5) = 20{,}160\pi - 12{,}000\pi = 8160\pi \text{ in}^3/\text{s}$.

31. (a) $V = \ell w h$, so by the Chain Rule,

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt}\\ &= 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6 \text{ m}^3/\text{s}\end{aligned}$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w+h)\frac{d\ell}{dt} + 2(\ell+h)\frac{dw}{dt} + 2(\ell+w)\frac{dh}{dt}\\ &= 2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10 \text{ m}^2/\text{s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \;\Rightarrow\; 2L\dfrac{dL}{dt} = 2\ell\dfrac{d\ell}{dt} + 2w\dfrac{dw}{dt} + 2h\dfrac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \;\Rightarrow$ $dL/dt = 0$ m/s.

32. $I = \dfrac{V}{R} \;\Rightarrow$

$$\begin{aligned}\frac{dI}{dt} &= \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{I}{R}\frac{dR}{dt}\\ &= \tfrac{1}{400}(-0.01) - \tfrac{0.08}{400}(0.03) = -0.000031 \text{ A/s}\end{aligned}$$

33. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$, $\dfrac{dV}{dt} = 8.31\left[\dfrac{0.15}{20} - \dfrac{(0.05)(320)}{400}\right] \approx -0.27$ L/s.

34. Let x and y be the respective distances of car A and car B from the intersection and let z be the distance between the two cars. Then $dx/dt = -90$, $dy/dt = -80$ and $z^2 = x^2 + y^2$. When $x = 0.3$ and $y = 0.4$, $z = \sqrt{0.25} = 0.5$ and $2z\,(dz/dt) = 2x\,(dx/dt) + 2y\,(dy/dt)$ or $dz/dt = 0.6(-90) + 0.8(-80) = -118$ km/h.

35. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}\,r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$\left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$.

36. By the Chain Rule, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}e^s\cos t + \dfrac{\partial u}{\partial y}e^s\sin t$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}(-e^s\sin t) + \dfrac{\partial u}{\partial y}e^s\cos t$. Then

$\left(\dfrac{\partial u}{\partial s}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t + 2\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}e^{2s}\cos t\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$ and

$\left(\dfrac{\partial u}{\partial t}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t - 2\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}e^{2s}\cos t\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$. Thus

$\left[\left(\dfrac{\partial u}{\partial s}\right)^2 + \left(\dfrac{\partial u}{\partial t}\right)^2\right]e^{-2s} = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2$.

41. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\,2s + \dfrac{\partial z}{\partial y}\,2r$. Then

$$\begin{aligned}
\frac{\partial^2 z}{\partial r\,\partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right) \\
&= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}\,2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\,2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}\,2s + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\,2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}\,2r + \frac{\partial z}{\partial y}\,2 \\
&= 4rs\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\,\partial x}\,4s^2 + 0 + 4rs\,\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\,\partial y}\,4r^2 + 2\,\frac{\partial z}{\partial y}
\end{aligned}$$

By the continuity of the partials, $\dfrac{\partial^2 z}{\partial r\,\partial s} = 4rs\,\dfrac{\partial^2 z}{\partial x^2} + 4rs\,\dfrac{\partial^2 z}{\partial y^2} + \left(4r^2 + 4s^2\right)\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y}$.

42. By the Chain Rule,

(a) $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$

(b) $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$

(c)
$$\begin{aligned}
\frac{\partial^2 z}{\partial r\,\partial \theta} &= \frac{\partial^2 z}{\partial \theta\,\partial r} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) \\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right) \\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial \theta}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial \theta} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial \theta} \\
&= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(-r\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} \\
&\qquad + \sin\theta\left(r\cos\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y}\right) \\
&= -\sin\theta\,\frac{\partial z}{\partial x} - r\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x} + \cos\theta\,\frac{\partial z}{\partial y} \\
&\qquad + r\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x} \\
&= \cos\theta\,\frac{\partial z}{\partial y} - \sin\theta\,\frac{\partial z}{\partial x} + r\cos\theta\sin\theta\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right) + r\left(\cos^2\theta - \sin^2\theta\right)\frac{\partial^2 z}{\partial y\,\partial x}
\end{aligned}$$

43. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$. Then

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right) \\
&= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

and

$$\frac{\partial^2 z}{\partial\theta^2} = -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}\,r\cos\theta\right)$$
$$-r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2}\,r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta)\right)$$
$$= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}$$

Thus

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 z}{\partial x^2} + (\sin^2\theta + \cos^2\theta)\,\frac{\partial^2 z}{\partial y^2} - \frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x}$$
$$-\frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)$$
$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}$$

44. (a) $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$. Then

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$
$$= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$
$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$
$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

(b)

$$\frac{\partial^2 z}{\partial s\,\partial t} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$
$$= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial s}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t}$$
$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x\,\partial y}\left(\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}$$

45. $F(x,y,z) = 0$ is assumed to define z as a function of x and y, that is, $z = f(x,y)$. So by (7), $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ since $F_z \neq 0$. Similarly, it is assumed that $F(x,y,z) = 0$ defines x as a function of y and z, that is $x = h(y,z)$. Then $F(h(y,z),y,z) = 0$ and by the Chain Rule, $F_x\dfrac{\partial x}{\partial y} + F_y\dfrac{\partial y}{\partial y} + F_z\dfrac{\partial z}{\partial y} = 0$. But $\dfrac{\partial z}{\partial y} = 0$ and $\dfrac{\partial y}{\partial y} = 1$, so $F_x\dfrac{\partial x}{\partial y} + F_y = 0 \;\Rightarrow\; \dfrac{\partial x}{\partial y} = -\dfrac{F_y}{F_x}$. A similar calculation shows that $\dfrac{\partial y}{\partial z} = -\dfrac{F_z}{F_y}$. Thus

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = \left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right) = -1.$$