3. a. Let H stand for "Healthy" and I stand for "Ill." Then the students' conditions are given by the table

From: H	From: I	To:
.95	.45	H
.05	.55	I

so the stochastic matrix is $P = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$.

b. Since 20% of the students are ill on Monday, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$. For Tuesday's percentages, we calculate $\mathbf{x}_1$; for Wednesday's percentages, we calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .85 \\ .15 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .85 \\ .15 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$$

Thus 15% of the students are ill on Tuesday, and 12.5% are ill on Wednesday.

c. Since the student is well today, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .925 \\ .075 \end{bmatrix}$$

Thus the probability that the student is well two days from now is .925.

4. a. Let G stand for good weather, I for indifferent weather, and B for bad weather. Then the change in the weather is given by the table

From: G	From: I	From: B	To:
.6	.4	.4	G
.3	.3	.5	I
.1	.3	.1	B

so the stochastic matrix is $P = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$.

b. The initial state vector is $\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_1$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

12. From Exercise 2, $P=\begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.5 & .25 & .25 \\ .25 & -.5 & .25 \\ .25 & .25 & -.5 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by ro

reducing the augmented matrix gives

$$\begin{bmatrix} -.5 & .25 & .25 & 0 \\ .25 & -.5 & .25 & 0 \\ .25 & .25 & -.5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ sum to 3, multiply by 1/3

obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}=\begin{bmatrix} .333 \\ .333 \\ .333 \end{bmatrix}$. Thus in the long run each food will be preferred

equally.

13. a. From Exercise 3, $P=\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.05 & .45 \\ .05 & -.45 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reduci

the augmented matrix gives

$$\begin{bmatrix} -.05 & .45 & 0 \\ .05 & -.45 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=x_2\begin{bmatrix} 9 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$ sum to 10, multiply by

to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} 9/10 \\ 1/10 \end{bmatrix}=\begin{bmatrix} .9 \\ .1 \end{bmatrix}$.

b. After many days, a specific student is ill with probability .1, and it does not matter whether that student is ill today or not.

14. From Exercise 4, $P=\begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.4 & .4 & .4 \\ .3 & -.7 & .5 \\ .1 & .3 & -.9 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reduci

the augmented matrix gives

$$\begin{bmatrix} -.4 & .4 & .4 & 0 \\ .3 & -.7 & .5 & 0 \\ .1 & .3 & -.9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ sum to 6, multiply by 1/6 to

obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix}\approx\begin{bmatrix} .5 \\ .333 \\ .167 \end{bmatrix}$. Thus in the long run the chance that a day has good

weather is 50%.

5.3 SOLUTIONS

1. $P=\begin{bmatrix}5&7\\2&3\end{bmatrix}, D=\begin{bmatrix}2&0\\0&1\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute $P^{-1}=\begin{bmatrix}3&-7\\-2&5\end{bmatrix}, D^4=\begin{bmatrix}16&0\\0&1\end{bmatrix}$,

and $A^4=\begin{bmatrix}5&7\\2&3\end{bmatrix}\begin{bmatrix}16&0\\0&1\end{bmatrix}\begin{bmatrix}3&-7\\-2&5\end{bmatrix}=\begin{bmatrix}226&-525\\90&-209\end{bmatrix}$

2. $P=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}, D=\begin{bmatrix}1&0\\0&1/2\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute

$P^{-1}=\begin{bmatrix}5&3\\3&2\end{bmatrix}, D^4=\begin{bmatrix}1&0\\0&1/16\end{bmatrix}$, and $A^4=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}\begin{bmatrix}1&0\\0&1/16\end{bmatrix}\begin{bmatrix}5&3\\3&2\end{bmatrix}=\frac{1}{16}\begin{bmatrix}151&90\\-225&-134\end{bmatrix}$

3. $A^k=PD^kP^{-1}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}a^k&0\\0&b^k\end{bmatrix}\begin{bmatrix}1&0\\-3&1\end{bmatrix}=\begin{bmatrix}a^k&0\\3a^k-3b^k&b^k\end{bmatrix}$.

4. $A^k=PD^kP^{-1}=\begin{bmatrix}3&4\\1&1\end{bmatrix}\begin{bmatrix}2^k&0\\0&1^k\end{bmatrix}\begin{bmatrix}-1&4\\1&-3\end{bmatrix}=\begin{bmatrix}4-3\cdot2^k&12\cdot2^k-12\\1-2^k&4\cdot2^k-3\end{bmatrix}$.

5. By the Diagonalization Theorem, eigenvectors form the columns of the left factor, and they correspond respectively to the eigenvalues on the diagonal of the middle factor.

$$\lambda=5:\begin{bmatrix}1\\1\\1\end{bmatrix};\lambda=1:\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\0\end{bmatrix}$$

6. As in Exercise 5, inspection of the factorization gives:

$$\lambda=4:\begin{bmatrix}-1\\2\\0\end{bmatrix};\lambda=5:\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$$

7. Since A is triangular, its eigenvalues are obviously ±1.

<u>For $\lambda=1$</u>: $A-1I=\begin{bmatrix}0&0\\6&-2\end{bmatrix}$. The equation $(A-1I)\mathbf{x}=\mathbf{0}$ amounts to $6x_1-2x_2=0$, so $x_1=(1/3)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}1/3\\1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\3\end{bmatrix}$.

<u>For $\lambda=-1$</u>: $A+1I=\begin{bmatrix}2&0\\6&0\end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $2x_1=0$, so $x_1=0$ with x_2 free. The general solution is $x_2\begin{bmatrix}0\\1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}0\\1\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=\begin{bmatrix}\mathbf{v}_1&\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}1&0\\3&1\end{bmatrix}$. Then set $D=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

8. Since A is triangular, its only eigenvalue is obviously 5.

<u>For $\lambda = 5$</u>: $A - 5I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The equation $(A-5I)\mathbf{x} = \mathbf{0}$ amounts to $x_2 = 0$, so $x_2 = 0$ with x_1 free. The general solution is $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable.

9. To find the eigenvalues of A, compute its characteristic polynomial:

$$\det(A - \lambda I) = \det\begin{bmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{bmatrix} = (3-\lambda)(5-\lambda) - (-1)(1) = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2$$

Thus the only eigenvalue of A is 4.

<u>For $\lambda = 4$</u>: $A - 4I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. The equation $(A-4I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + x_2 = 0$, so $x_1 = -x_2$ with x_2 free. The general solution is $x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable.

10. To find the eigenvalues of A, compute its characteristic polynomial:

$$\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - (3)(4) = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$$

Thus the eigenvalues of A are 5 and -2.

<u>For $\lambda = 5$</u>: $A - 5I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$. The equation $(A-5I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 - x_2 = 0$, so $x_1 = x_2$ with x_2 free. The general solution is $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

<u>For $\lambda = -2$</u>: $A + 2I = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$. The equation $(A+1I)\mathbf{x} = \mathbf{0}$ amounts to $4x_1 + 3x_2 = 0$, so $x_1 = (-3/4)x_2$ with x_2 free. The general solution is $x_2 \begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$. Then set $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

11. The eigenvalues of A are given to be 1, 2, and 3.

<u>For $\lambda = 3$</u>: $A - 3I = \begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 3I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

For $\lambda = 2$: $A-2I=\begin{bmatrix}-3&4&-2\\-3&2&0\\-3&1&1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-2/3&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}2/3\\1\\1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}2\\3\\3\end{bmatrix}$.

For $\lambda = 1$: $A-I=\begin{bmatrix}-2&4&-2\\-3&3&0\\-3&1&2\end{bmatrix}$, and row reducing $\begin{bmatrix}A-1I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}1&2&1\\3&3&1\\4&3&1\end{bmatrix}$. Then set $D=\begin{bmatrix}3&0&0\\0&2&0\\0&0&1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

12. The eigenvalues of A are given to be 2 and 8.

For $\lambda = 8$: $A-8I=\begin{bmatrix}-4&2&2\\2&-4&2\\2&2&-4\end{bmatrix}$, and row reducing $\begin{bmatrix}A-8I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

For $\lambda = 2$: $A-2I=\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}1&-1&-1\\1&1&0\\1&0&1\end{bmatrix}$. Then set $D=\begin{bmatrix}8&0&0\\0&2&0\\0&0&2\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

$\begin{bmatrix} -3/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$,

answer differs from

ew order of the

$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The

$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$,

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. The

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$,

From $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ construct $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

21. **a.** False. The symbol D does not automatically denote a diagonal matrix.

b. True. See the remark after the statement of the Diagonalization Theorem.

c. False. The 3×3 matrix in Example 4 has 3 eigenvalues, counting multiplicities, but it is not diagonalizable.

d. False. Invertibility depends on 0 not being an eigenvalue. (See the Invertible Matrix Theorem.) A diagonalizable matrix may or may not have 0 as an eigenvalue. See Examples 3 and 5 for both possibilities.

22. **a.** False. The n eigenvectors must be linearly independent. See the Diagonalization Theorem.

b. False. The matrix in Example 3 is diagonalizable, but it has only 2 distinct eigenvalues. (The statement given is the *converse* of Theorem 6.)

c. True. This follows from $AP = PD$ and formulas (1) and (2) in the proof of the Diagonalization Theorem.

d. False. See Example 4. The matrix there is invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.

23. A is diagonalizable because you know that five linearly independent eigenvectors exist: three in the three-dimensional eigenspace and two in the two-dimensional eigenspace. Theorem 7 guarantees that the set of all five eigenvectors is linearly independent.

24. No, by Theorem 7(b). Here is an explanation that does not appeal to Theorem 7: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that span the two one-dimensional eigenspaces. If $\mathbf{v}$ is any other eigenvector, then it belongs to one of the eigenspaces and hence is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. So there cannot exist three linearly independent eigenvectors. By the Diagonalization Theorem, A cannot be diagonalizable.

25. Let $\{\mathbf{v}_1\}$ be a basis for the one-dimensional eigenspace, let $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis for the two-dimensional eigenspace, and let $\mathbf{v}_4$ be any eigenvector in the remaining eigenspace. By Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. Since A is 4×4, the Diagonalization Theorem shows that A is diagonalizable.

26. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 24 can also be given.

27. If A is diagonalizable, then $A = PDP^{-1}$ for some invertible P and diagonal D. Since A is invertible, 0 is not an eigenvalue of A. So the diagonal entries in D (which are eigenvalues of A) are not zero, and D is invertible. By the theorem on the inverse of a product,

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$$

Since D^{-1} is obviously diagonal, A^{-1} is diagonalizable.

28. If A has n linearly independent eigenvectors, then by the Diagonalization Theorem, $A = PDP^{-1}$ for some invertible P and diagonal D. Using properties of transposes,

$$\begin{aligned} A^T &= (PDP^{-1})^T = (P^{-1})^T D^T P^T \\ &= (P^T)^{-1} DP^T = QDQ^{-1} \end{aligned}$$

where $Q = (P^T)^{-1}$. Thus A^T is diagonalizable. By the Diagonalization Theorem, the columns of Q are n linearly independent eigenvectors of A^T.

29. The diagonal entries in D_1 are reversed from those in D. So interchange the (eigenvector) columns of P to make them correspond properly to the eigenvalues in D_1. In this case,

$$P_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \text{ and } D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Although the first column of P must be an eigenvector corresponding to the eigenvalue 3, there is nothing to prevent us from selecting some multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, say $\begin{bmatrix} -3 \\ 6 \end{bmatrix}$, and letting $P_2 = \begin{bmatrix} -3 & 1 \\ 6 & -1 \end{bmatrix}$. We now have three different factorizations or "diagonalizations" of A:

$$A = PDP^{-1} = P_1D_1P_1^{-1} = P_2D_1P_2^{-1}$$

30. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar unequal to 1.

31. For a 2×2 matrix A to be invertible, its eigenvalues must be nonzero. A first attempt at a construction might be something such as $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, whose eigenvalues are 2 and 4. Unfortunately, a 2×2 matrix with two distinct eigenvalues is diagonalizable (Theorem 6). So, adjust the construction to $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, which works. In fact, any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ has the desired properties when a and b are nonzero. The eigenspace for the eigenvalue a is one-dimensional, as a simple calculation shows, and there is no other eigenvalue to produce a second eigenvector.

32. Any 2×2 matrix with two distinct eigenvalues is diagonalizable, by Theorem 6. If one of those eigenvalues is zero, then the matrix will not be invertible. Any matrix of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has the desired properties when a and b are nonzero. The number a must be nonzero to make the matrix diagonalizable; b must be nonzero to make the matrix not diagonal. Other solutions are $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$.

3. $A=\begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}$. The characteristic polynomial is $\lambda^2-4\lambda+13$, so the eigenvalues of A are $\lambda=\frac{4\pm\sqrt{-36}}{2}=2\pm 3i$.

<u>For $\lambda=2+3i$:</u> $A-(2+3i)I=\begin{bmatrix}-1-3i & 5\\ -2 & 1-3i\end{bmatrix}$. The equation $(A-(2+3i)I)\mathbf{x}=\mathbf{0}$ amounts to $-2x_1+(1-3i)x_2=0$, so $x_1=\frac{1-3i}{2}x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}1-3i\\ 2\end{bmatrix}$.

<u>For $\lambda=2-3i$:</u> A basis vector for the eigenspace is $\mathbf{v}_2=\bar{\mathbf{v}}_1=\begin{bmatrix}1+3i\\ 2\end{bmatrix}$.

4. $A=\begin{bmatrix}5 & -2\\ 1 & 3\end{bmatrix}$. The characteristic polynomial is $\lambda^2-8\lambda+17$, so the eigenvalues of A are $\lambda=\frac{8\pm\sqrt{-4}}{2}=4\pm i$.

<u>For $\lambda=4+i$:</u> $A-(4+i)I=\begin{bmatrix}1-i & -2\\ 1 & -1-i\end{bmatrix}$. The equation $(A-(4+i)I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+(-1-i)x_2=0$, so $x_1=(1+i)x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}1+i\\ 1\end{bmatrix}$.

<u>For $\lambda=4-i$:</u> A basis vector for the eigenspace is $\mathbf{v}_2=\bar{\mathbf{v}}_1=\begin{bmatrix}1-i\\ 1\end{bmatrix}$.

5. $A=\begin{bmatrix}0 & 1\\ -8 & 4\end{bmatrix}$. The characteristic polynomial is $\lambda^2-4\lambda+8$, so the eigenvalues of A are $\lambda=\frac{4\pm\sqrt{-16}}{2}=2\pm 2i$.

<u>For $\lambda=2+2i$:</u> $A-(2+2i)I=\begin{bmatrix}-2-2i & 1\\ -8 & 2-2i\end{bmatrix}$. The equation $(A-(2+2i)I)\mathbf{x}=\mathbf{0}$ amounts to $(-2-2i)x_1+x_2=0$, so $x_2=(2+2i)x_1$ with x_1 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}1\\ 2+2i\end{bmatrix}$.

<u>For $\lambda=2-2i$:</u> A basis vector for the eigenspace is $\mathbf{v}_2=\bar{\mathbf{v}}_1=\begin{bmatrix}1\\ 2-2i\end{bmatrix}$.

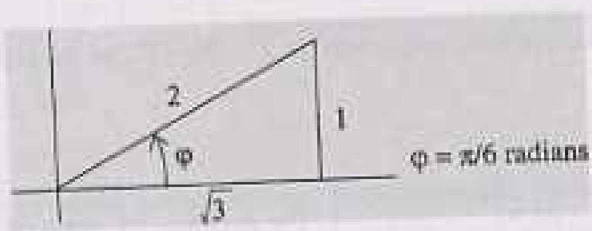
6. $A=\begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix}$. The characteristic polynomial is $\lambda^2-8\lambda+25$, so the eigenvalues of A are $\lambda=\frac{8\pm\sqrt{-36}}{2}=4\pm 3i$.

<u>For $\lambda = 4+3i$:</u> $A-(4+3i)I = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$. The equation $(A-(4+3i)I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + ix_2 = 0$, so $x_1 = -ix_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

<u>For $\lambda = 4-3i$:</u> A basis vector for the eigenspace is $\mathbf{v}_2 = \bar{\mathbf{v}}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

7. $A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$. From Example 6, the eigenvalues are $\sqrt{3} \pm i$. The scale factor for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $r = |\lambda| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$. For the angle of rotation, plot the point $(a,b) = (\sqrt{3}, 1)$ in the xy-plane and use trigonometry:

$$\varphi = \arctan(b/a) = \arctan(1/\sqrt{3}) = \pi/6 \text{ radians.}$$



Note: Your students will want to know whether you permit them on an exam to omit calculations for a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and simply write the eigenvalues $a \pm bi$. A similar question may arise about the corresponding eigenvectors, $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$, which are announced in the Practice Problem. Students may have trouble keeping track of the correspondence between eigenvalues and eigenvectors.

8. $A = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$. From Example 6, the eigenvalues are $\sqrt{3} \pm 3i$. The scale factor for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $r = |\lambda| = \sqrt{(\sqrt{3})^2 + 3^2} = 2\sqrt{3}$. From trigonometry, the angle of rotation φ is $\arctan(b/a) = \arctan(-3/\sqrt{3}) = -\pi/3$ radians.

9. $A = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$. From Example 6, the eigenvalues are $-\sqrt{3}/2 \pm (1/2)i$. The scale factor for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $r = |\lambda| = \sqrt{(-\sqrt{3}/2)^2 + (1/2)^2} = 1$. From trigonometry, the angle of rotation φ is $\arctan(b/a) = \arctan((-1/2)/(-\sqrt{3}/2)) = -5\pi/6$ radians.

15. $A=\begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}$. From Exercise 3, the eigenvalues of A are $\lambda = 2 \pm 3i$, and the eigenvector

$\mathbf{v}=\begin{bmatrix}1+3i\\ 2\end{bmatrix}$ corresponds to $\lambda = 2-3i$. By Theorem 9, $P=[\text{Re } \mathbf{v} \;\; \text{Im } \mathbf{v}]=\begin{bmatrix}1 & 3\\ 2 & 0\end{bmatrix}$ and

$$C=P^{-1}AP=\frac{1}{6}\begin{bmatrix}0 & -3\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 0\end{bmatrix}=\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix}$$

16. $A=\begin{bmatrix}5 & -2\\ 1 & 3\end{bmatrix}$. From Exercise 4, the eigenvalues of A are $\lambda = 4 \pm i$, and the eigenvector

$\mathbf{v}=\begin{bmatrix}1-i\\ 1\end{bmatrix}$ corresponds to $\lambda = 4-i$. By Theorem 9, $P=[\text{Re } \mathbf{v} \;\; \text{Im } \mathbf{v}]=\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$ and

$$C=P^{-1}AP=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}5 & -2\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}4 & -1\\ 1 & 4\end{bmatrix}$$

17. $A=\begin{bmatrix}1 & -.8\\ 4 & -2.2\end{bmatrix}$. The characteristic polynomial is $\lambda^2+1.2\lambda+1$, so the eigenvalues of A are $\lambda = -.6 \pm .8i$.

To find an eigenvector corresponding to $-.6-.8i$, we compute

$$A-(-.6-.8i)I=\begin{bmatrix}1.6+.8i & -.8\\ 4 & -1.6+.8i\end{bmatrix}$$

The equation $(A-(-.6-.8i)I)\mathbf{x}=\mathbf{0}$ amounts to $4x_1+(-1.6+.8i)x_2=0$, so $x_1=((2-i)/5)x_2$ with x_2 free. A nice eigenvector corresponding to $-.6-.8i$ is thus $\mathbf{v}=\begin{bmatrix}2-i\\ 5\end{bmatrix}$. By Theorem 9,

$$P=[\text{Re } \mathbf{v} \;\; \text{Im } \mathbf{v}]=\begin{bmatrix}2 & -1\\ 5 & 0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{5}\begin{bmatrix}0 & 1\\ -5 & 2\end{bmatrix}\begin{bmatrix}1 & -.8\\ 4 & -2.2\end{bmatrix}\begin{bmatrix}2 & -1\\ 5 & 0\end{bmatrix}=\begin{bmatrix}-.6 & -.8\\ .8 & -.6\end{bmatrix}$$

18. $A=\begin{bmatrix}1 & -1\\ .4 & .6\end{bmatrix}$. The characteristic polynomial is $\lambda^2-1.6\lambda+1$, so the eigenvalues of A are $\lambda = .8 \pm .6i$. To find an eigenvector corresponding to $.8-.6i$, we compute

$$A-(.8-.6i)I=\begin{bmatrix}.2+.6i & -1\\ .4 & -.2+.6i\end{bmatrix}$$

The equation $(A-(.8-.6i)I)\mathbf{x}=\mathbf{0}$ amounts to $.4x_1+(-.2+.6i)x_2=0$, so $x_1=((1-3i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.8-.6i$ is thus $\mathbf{v}=\begin{bmatrix}1-3i\\ 2\end{bmatrix}$. By Theorem 9,

$$P=[\text{Re } \mathbf{v} \;\; \text{Im } \mathbf{v}]=\begin{bmatrix}1 & -3\\ 2 & 0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{6}\begin{bmatrix}0 & 3\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ .4 & .6\end{bmatrix}\begin{bmatrix}1 & -3\\ 2 & 0\end{bmatrix}=\begin{bmatrix}.8 & -.6\\ .6 & .8\end{bmatrix}$$

19. $A=\begin{bmatrix}1.52 & -.7\\ .56 & .4\end{bmatrix}$. The characteristic polynomial is $\lambda^2-1.92\lambda+1$, so the eigenvalues of A are $\lambda=.96\pm.28i$. To find an eigenvector corresponding to $.96-.28i$, we compute

$$A-(.96-.28i)I=\begin{bmatrix}.56+.28i & -.7\\ .56 & -.56+.28i\end{bmatrix}$$

The equation $(A-(.96-.28i)I)\mathbf{x}=\mathbf{0}$ amounts to $.56x_1+(-.56+.28i)x_2=0$, so $x_1=((2-i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.96-.28i$ is thus $\mathbf{v}=\begin{bmatrix}2-i\\ 2\end{bmatrix}$. By Theorem 9,

$$P=[\operatorname{Re}\mathbf{v}\quad \operatorname{Im}\mathbf{v}]=\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{2}\begin{bmatrix}0 & 1\\ -2 & 2\end{bmatrix}\begin{bmatrix}1.52 & -.7\\ .56 & .4\end{bmatrix}\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}.96 & -.28\\ .28 & .96\end{bmatrix}$$

20. $A=\begin{bmatrix}-1.64 & -2.4\\ 1.92 & 2.2\end{bmatrix}$. The characteristic polynomial is $\lambda^2-.56\lambda+1$, so the eigenvalues of A are $\lambda=.28\pm.96i$. To find an eigenvector corresponding to $.28-.96i$, we compute

$$A-(.28-.96i)I=\begin{bmatrix}-1.92+.96i & -2.4\\ 1.92 & 1.92+.96i\end{bmatrix}$$

The equation $(A-(.28-.96i)I)\mathbf{x}=\mathbf{0}$ amounts to $1.92x_1+(1.92+.96i)x_2=0$, so $x_1=((-2-i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.28-.96i$ is thus $\mathbf{v}=\begin{bmatrix}-2-i\\ 2\end{bmatrix}$. By Theorem 9,

$$P=[\operatorname{Re}\mathbf{v}\quad \operatorname{Im}\mathbf{v}]=\begin{bmatrix}-2 & -1\\ 2 & 0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{2}\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}-1.64 & -2.4\\ 1.92 & 2.2\end{bmatrix}\begin{bmatrix}-2 & -1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}.28 & -.96\\ .96 & .28\end{bmatrix}$$

21. The first equation in (2) is $(-.3+.6i)x_1-.6x_2=0$. We solve this for x_2 to find that $x_2=((-.3+.6i)/.6)x_1=((-1+2i)/2)x_1$. Letting $x_1=2$, we find that $\mathbf{y}=\begin{bmatrix}2\\ -1+2i\end{bmatrix}$ is an eigenvector for the matrix A. Since $\mathbf{y}=\begin{bmatrix}2\\ -1+2i\end{bmatrix}=\frac{-1+2i}{5}\begin{bmatrix}-2-4i\\ 5\end{bmatrix}=\frac{-1+2i}{5}\mathbf{v}_1$ the vector $\mathbf{y}$ is a complex multiple of the vector $\mathbf{v}_1$ used in Example 2.

22. Since $A(\mu\mathbf{x})=\mu(A\mathbf{x})=\mu(\lambda\mathbf{x})=\lambda(\mu\mathbf{x})$, $\mu\mathbf{x}$ is an eigenvector of A.

23. (a) properties of conjugates and the fact that $\bar{\mathbf{x}}^T=\overline{\mathbf{x}^T}$

(b) $\overline{A\mathbf{x}}=A\bar{\mathbf{x}}$ and A is real

(c) $\mathbf{x}^TA\bar{\mathbf{x}}$ is a scalar and hence may be viewed as a 1×1 matrix

(d) properties of transposes

(e) $A^T=A$ and the definition of q

24. $\bar{\mathbf{x}}^TA\mathbf{x}=\bar{\mathbf{x}}^T(\lambda\mathbf{x})=\lambda\cdot\bar{\mathbf{x}}^T\mathbf{x}$ because $\mathbf{x}$ is an eigenvector. It is easy to see that $\bar{\mathbf{x}}^T\mathbf{x}$ is real (and positive) because $\bar{z}z$ is nonnegative for every complex number z. Since $\bar{\mathbf{x}}^TA\mathbf{x}$ is real, by Exercise 23, so is λ. Next, write $\mathbf{x}=\mathbf{u}+i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are real vectors. Then

$$A\mathbf{x}=A(\mathbf{u}+i\mathbf{v})=A\mathbf{u}+iA\mathbf{v}\quad\text{and}\quad\lambda\mathbf{x}=\lambda\mathbf{u}+i\lambda\mathbf{v}$$

The real part of $A\mathbf{x}$ is $A\mathbf{u}$ because the entries in A, $\mathbf{u}$, and $\mathbf{v}$ are all real. The real part of $\lambda\mathbf{x}$ is $\lambda\mathbf{u}$ because λ and the entries in $\mathbf{u}$ and $\mathbf{v}$ are real. Since $A\mathbf{x}$ and $\lambda\mathbf{x}$ are equal, their real parts are equal, too. (Apply the corresponding statement about complex numbers to each entry of $A\mathbf{x}$.) Thus $A\mathbf{u}=\lambda\mathbf{u}$, which shows that the real part of $\mathbf{x}$ is an eigenvector of A.

25. Write $\mathbf{x}=\text{Re}\,\mathbf{x}+i(\text{Im}\,\mathbf{x})$, so that $A\mathbf{x}=A(\text{Re}\,\mathbf{x})+iA(\text{Im}\,\mathbf{x})$. Since A is real, so are $A(\text{Re}\,\mathbf{x})$ and $A(\text{Im}\,\mathbf{x})$. Thus $A(\text{Re}\,\mathbf{x})$ is the real part of $A\mathbf{x}$ and $A(\text{Im}\,\mathbf{x})$ is the imaginary part of $A\mathbf{x}$.

26. a. If $\lambda=a-bi$, then

$$A\mathbf{v}=\lambda\mathbf{v}=(a-bi)(\text{Re}\,\mathbf{v}+i\,\text{Im}\,\mathbf{v})$$
$$=\underbrace{(a\,\text{Re}\,\mathbf{v}+b\,\text{Im}\,\mathbf{v})}_{\text{Re}\,A\mathbf{v}}+i\underbrace{(a\,\text{Im}\,\mathbf{v}-b\,\text{Re}\,\mathbf{v})}_{\text{Im}\,A\mathbf{v}}$$

By Exercise 25,

$$A(\text{Re}\,\mathbf{v})=\text{Re}\,A\mathbf{v}=a\,\text{Re}\,\mathbf{v}+b\,\text{Im}\,\mathbf{v}$$
$$A(\text{Im}\,\mathbf{v})=\text{Im}\,A\mathbf{v}=-b\,\text{Re}\,\mathbf{v}+a\,\text{Im}\,\mathbf{v}$$

b. Let $P=[\text{Re}\,\mathbf{v}\quad\text{Im}\,\mathbf{v}]$. By (a),

$$A(\text{Re}\,\mathbf{v})=P\begin{bmatrix}a\\b\end{bmatrix},\ A(\text{Im}\,\mathbf{v})=P\begin{bmatrix}-b\\a\end{bmatrix}$$

So

$$AP=[A(\text{Re}\,\mathbf{v})\quad A(\text{Im}\,\mathbf{v})]$$
$$=\left[P\begin{bmatrix}a\\b\end{bmatrix}\ P\begin{bmatrix}-b\\a\end{bmatrix}\right]=P\begin{bmatrix}a&-b\\b&a\end{bmatrix}=PC$$

27. [M] $A=\begin{bmatrix}.7&1.1&2.0&1.7\\-2.0&-4.0&-8.6&-7.4\\0&-.5&-1.0&-1.0\\1.0&2.8&6.0&5.3\end{bmatrix}$

```
ev=eig(A)=(.2+.5i,.2-.5i,.3+.1i,.3-.1i)
```

For $\lambda=.2-.5i$, an eigenvector is

```
nulbasis(A-ev(2)*eye(4)) =
 0.5000 - 0.5000i
-2.0000 + 0.0000i
 0.0000 - 0.0000i
 1.0000
```

so that $\mathbf{v}_1=\begin{bmatrix}.5-.5i\\-2\\0\\1\end{bmatrix}$

For $\lambda=.3-.1i$, an eigenvector is

```
nulbasis(A-ev(4)*eye(4))=
-0.5000 - 0.0000i
```