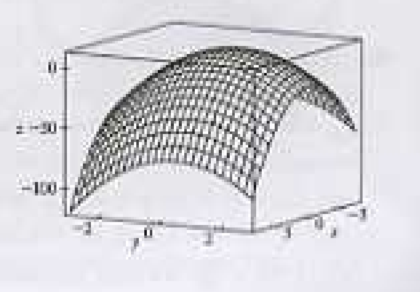
To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \Rightarrow f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3\left(x^2\right)^2 - 3x = 0 \Rightarrow 3x(x^3 - 1) = 0 \Rightarrow x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm 1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

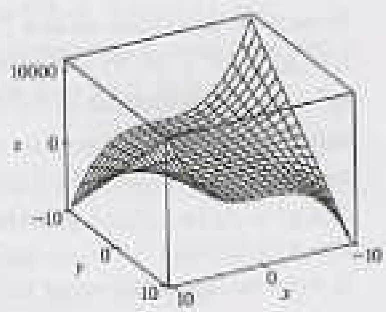
To confirm our predictions, we have $f(x,y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x,y) = 3 - 3x^2$, $f_y(x,y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \Rightarrow x = \pm 1$ and $-4y + 4y^3 = 0 \Rightarrow y(y^2 - 1) = 0 \Rightarrow y = 0, \pm 1$. So our critical points are $(\pm 1, 0)$, $(\pm 1, \pm 1)$. The second partial derivatives are $f_{xx}(x,y) = -6x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2 - 4$, so $D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x$. We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D > 0$, $f_{xx} < 0 \Rightarrow f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D < 0 \Rightarrow f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D > 0$, $f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D > 0$, $f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1,-1)$

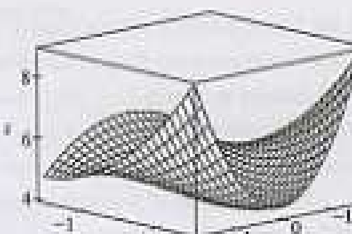
5. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2 \Rightarrow f_x = -2 - 2x$, $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is $\left(-1, \frac{1}{2}\right)$. $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and since $D\left(-1, \frac{1}{2}\right) = 16 > 0$ and $f_{xx}\left(-1, \frac{1}{2}\right) = -2 < 0$, $f\left(-1, \frac{1}{2}\right) = 11$ is a local maximum by the Second Derivatives Test.



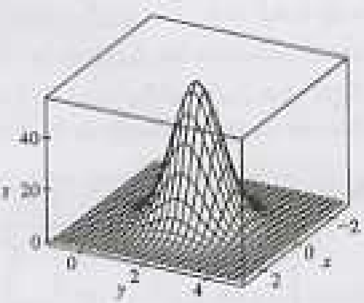
6. $f(x,y) = x^3y + 12x^2 - 8y \;\Rightarrow\; f_x = 3x^2y + 24x$, $f_y = x^3 - 8$, $f_{xx} = 6xy + 24$, $f_{xy} = 3x^2$, $f_{yy} = 0$. Then $f_y = 0$ implies $x = 2$, and substitution into $f_x = 0$ gives $12y + 48 = 0 \;\Rightarrow\; y = -4$. Thus, the only critical point is $(2, -4)$. $D(2,-4) = (-24)(0) - 12^2 = -144 < 0$, so $(2,-4)$ is a saddle point.



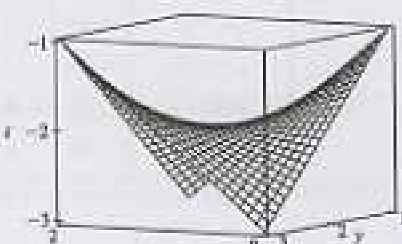
7. $f(x,y) = x^2 + y^2 + x^2y + 4 \;\Rightarrow\; f_x = 2x + 2xy$, $f_y = 2y + x^2$, $f_{xx} = 2 + 2y$, $f_{yy} = 2$, $f_{xy} = 2x$. Then $f_y = 0$ implies $y = -\frac{1}{2}x^2$, substituting into $f_x = 0$ gives $2x - x^3 = 0$ so $x = 0$ or $x = \pm\sqrt{2}$. Thus the critical points are $(0,0)$, $(\sqrt{2},-1)$ and $(-\sqrt{2},-1)$. Now $D(0,0) = 4$, $D(\sqrt{2},-1) = -8 = D(-\sqrt{2},-1)$, $f_{xx}(0,0) = 2$, $f_{xx}(\pm\sqrt{2},-1) = 0$.
Thus $f(0,0) = 4$ is a local minimum and $(\pm\sqrt{2},-1)$ are saddle points.



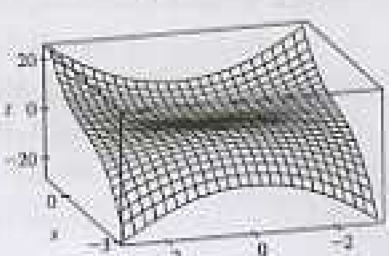
8. $f(x,y) = e^{4y-x^2-y^2} \;\Rightarrow\; f_x = -2xe^{4y-x^2-y^2}$, $f_y = (4-2y)e^{4y-x^2-y^2}$, $f_{xx} = (4x^2-2)e^{4y-x^2-y^2}$, $f_{xy} = -2x(4-2y)e^{4y-x^2-y^2}$, $f_{yy} = (4y^2 - 16y + 14)e^{4y-x^2-y^2}$. Then $f_x = 0$ and $f_y = 0$ implies $x = 0$ and $y = 2$, so the only critical point is $(0,2)$. $D(0,2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$ and $f_{xx}(0,2) = -2e^4 < 0$, so $f(0,2) = e^4$ is a local maximum.



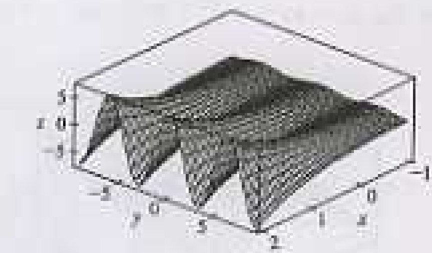
9. $f(x,y) = xy - 2x - y \;\Rightarrow\; f_x = y - 2$, $f_y = x - 1$, $f_{xx} = f_{yy} = 0$, $f_{xy} = 1$ and the only critical point is $(1,2)$. Now $D(1,2) = -1$, so $(1,2)$ is a saddle point and f has no local maximum or minimum.



10. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 \Rightarrow f_x = 6x^2 + y^2 + 10x$, $f_y = 2xy + 2y$, $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, $f_{xy} = 2y$. Then $f_y = 0$ implies $y = 0$ or $x = -1$. Substituting into $f_x = 0$ gives the critical points $(0,0)$, $(-\frac{5}{3},0)$, $(-1,\pm 2)$. Now $D(0,0) = 20 > 0$ and $f_{xx}(0,0) = 10 > 0$, so $f(0,0) = 0$ is a local minimum. Also $f_{xx}(-\frac{5}{3},0) < 0$, $D(-\frac{5}{3},0) > 0$, and $D(-1,\pm 2) < 0$. Hence $f(-\frac{5}{3},0) = \frac{125}{27}$ is a local maximum while $(-1,\pm 2)$ are saddle

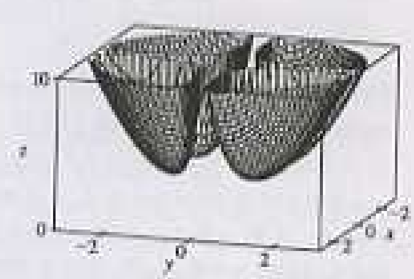


11. $f(x,y) = e^x \cos y \;\Rightarrow\; f_x = e^x \cos y$, $f_y = -e^x \sin y$. Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.

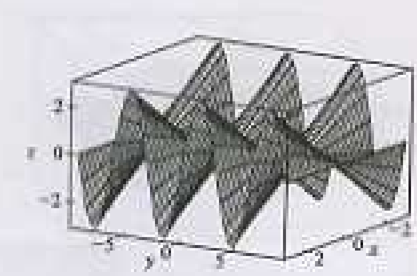


12. $f(x,y) = x^2 + y^2 + \dfrac{1}{x^2y^2} \;\Rightarrow\; f_x = 2x - 2x^{-3}y^{-2}$,

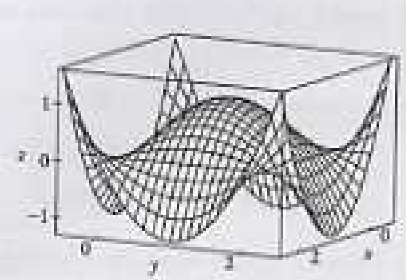
$f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$, $f_{xy} = 4x^{-3}y^{-3}$. Then $f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. Now $D(\pm 1, \pm 1) = D(\pm 1, \mp 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(\pm 1, \pm 1) = f(\pm 1, \mp 1) = 3$ are local minima.



13. $f(x,y) = x \sin y \;\Rightarrow\; f_x = \sin y$, $f_y = x \cos y$, $f_{xx} = 0$, $f_{yy} = -x \sin y$ and $f_{xy} = \cos y$. Then $f_x = 0$ if and only if $y = n\pi$, n an integer, and substituting into $f_y = 0$ requires $x = 0$ for each of these y-values. Thus the critical points are $(0, n\pi)$, n an integer. But $D(0, n\pi) = -\cos^2(n\pi) < 0$ so each critical point is a saddle point.

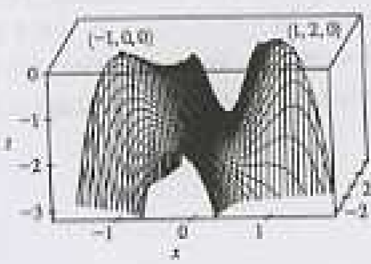


14. $f(x,y) = (2x - x^2)(2y - y^2) \;\Rightarrow\; f_x = (2 - 2x)(2y - y^2)$, $f_y = (2x - x^2)(2 - 2y)$, $f_{xx} = -2(2y - y^2)$, $f_{yy} = -2(2x - x^2)$ and $f_{xy} = (2 - 2x)(2 - 2y)$. Then $f_x = 0$ implies $x = 1$ or $y = 0$ or $y = 2$ and when $x = 1$, $f_y = 0$ implies $y = 1$, when $y = 0$, $f_y = 0$ implies $x = 0$ or $x = 2$ and when $y = 2$, $f_y = 0$ implies $x = 0$ or $x = 2$. Thus the critical points are $(1,1)$, $(0,0)$, $(2,0)$, $(0,2)$ and $(2,2)$. Now $D(0,0) = D(2,0) = D(0,2) = D(2,2) = -16$ so these critical points are saddle points, and $D(1,1) = 4$ with $f_{xx}(1,1) = -2$, so $f(1,1) = 1$ is a local maximum.



29. $f(x,y) = -(x^2-1)^2 - (x^2y-x-1)^2 \Rightarrow f_x(x,y) = -2(x^2-1)(2x) - 2(x^2y-x-1)(2xy-1)$ and $f_y(x,y) = -2(x^2y-x-1)\,x^2$. Setting $f_y(x,y)=0$ gives either $x=0$ or $x^2y-x-1=0$. There are no critical points for $x=0$, since $f_x(0,y)=-2$, so we set $x^2y-x-1=0 \Leftrightarrow y=\frac{x+1}{x^2}$ $(x\neq 0)$, so

$f_x\left(x,\frac{x+1}{x^2}\right) = -2\left(x^2-1\right)(2x) - 2\left(x^2\frac{x+1}{x^2}-x-1\right)\left(2x\frac{x+1}{x^2}-1\right) = -4x(x^2-1)$. Therefore $f_x(x,y) = f_y(x,y) = 0$ at the points $(1,2)$ and $(-1,0)$.

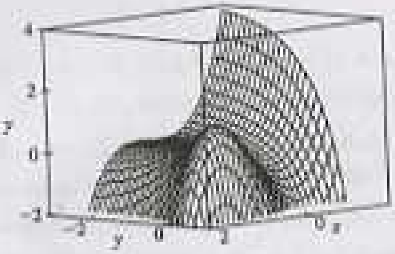


To classify these critical points, we calculate
$f_{xx}(x,y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$,
$f_{yy}(x,y) = -2x^4$, and $f_{xy}(x,y) = -8x^3y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate

$$D(-1,0) = f_{xx}(-1,0)\,f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = 16 > 0,$$

$f_{xx}(-1,0) = -10 < 0$, $D(1,2) = 16 > 0$, and $f_{xx}(1,2) = -26 < 0$, so both $(-1,0)$ and $(1,2)$ give local maxima.

30. $f(x,y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that (1) $f_x = 3e^y - 3x^2 = 0$ and (2) $f_y = 3xe^y - 3e^{3y} = 0$. From (1) we obtain $e^y = x^2$, and then (2) gives $3x^3 - 3x^6 = 0 \Rightarrow x = 1$ or 0, but only $x=1$ is valid, since $x=0$ makes (1) impossible. So substituting $x=1$ into (1) gives $y=0$, and the only critical point is $(1,0)$.



The Second Derivatives Test shows that this gives a local maximum, since $D(1,0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1,0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1,0) = 1$ is not an absolute maximum because, for instance, $f(-3,0) = 17$. This can also be seen from the graph.

31. Let d be the distance from $(2,1,-1)$ to any point (x,y,z) on the plane $x+y-z=1$, so $d = \sqrt{(x-2)^2+(y-1)^2+(z+1)^2}$ where $z = x+y-1$, and we minimize $d^2 = f(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2$. Then $f_x(x,y) = 2(x-2) + 2(x+y) = 4x+2y-4$, $f_y(x,y) = 2(y-1) + 2(x+y) = 2x+4y-2$. Solving $4x+2y-4=0$ and $2x+4y-2=0$ simultaneously gives $x=1$, $y=0$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x=1$, $y=0$ for which $d = \sqrt{(1-2)^2+(0-1)^2+(1+0)^2} = \sqrt{3}$.

42. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so
$C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2Vy^{-2}$,
$f_x = 0$ implies $y = 2V/(5x^2)$, $f_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the box which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

43. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\text{ m}^3$. Then
$f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$.
And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$.
Now $D(x, y) = [(2)(64{,}000)]^2 x^{-3}y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum.
Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

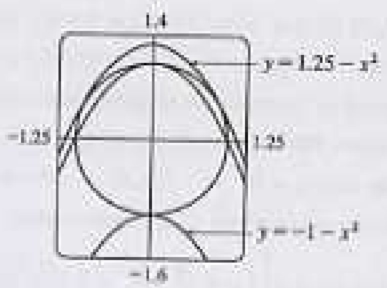
44. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving
$P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \geq 0$, $r \geq 0$, and $q + r \leq 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q, r) = 2 - 4q - 2r = 0$ and $P_r(q, r) = 2 - 4r - 2q = 0$. The first equation gives $r = 1 - 2q$, and substituting into the second equation we have $2 - 4(1 - 2q) - 2q = 0 \Rightarrow q = \frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $P\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0$, $0 \leq q \leq 1$, $P(q, r) = P(q, 0) = 2q - 2q^2$, $0 \leq q \leq 1$. This represents a parabola with maximum value $P\left(\frac{1}{2}, 0\right) = \frac{1}{2}$. On the segment $q = 0$, $0 \leq r \leq 1$ we have $P(0, r) = 2r - 2r^2$, $0 \leq r \leq 1$. This represents a parabola with maximum value $P\left(0, \frac{1}{2}\right) = \frac{1}{2}$. Finally, on the segment $q + r = 1$, $0 \leq q \leq 1$,
$P(q, r) = P(q, 1 - q) = 2q - 2q^2$, $0 \leq q \leq 1$ which has a maximum value of $P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q, r)$ on D is $\frac{2}{3}$.

45. Note that here the variables are m and b, and $f(m, b) = \sum_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then
$f_m = \sum_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum_{i=1}^{n} (x_i y_i - mx_i^2 - bx_i) = 0$ or $\sum_{i=1}^{n} x_i y_i = m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i$
and $f_b = \sum_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum_{i=1}^{n} y_i = m \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} b = m\left(\sum_{i=1}^{n} x_i\right) + nb$. Thus we have the two desired equations. Now $f_{mm} = \sum_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum_{i=1}^{n} 2x_i$. And $f_{mm}(m, b) > 0$ always and $D(m, b) = 4n\left(\sum_{i=1}^{n} x_i^2\right) - 4\left(\sum_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2\right] > 0$ always so the solutions of these two equations do indeed minimize $\sum_{i=1}^{n} d_i^2$.

11.8 Lagrange Multipliers

1. At the extreme values of f, the level curves of f just touch the curve $g(x, y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x, y) = c$ with the largest value of c which still intersects the curve $g(x, y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x, y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x, y) = 8$ to be about 59 and the minimum to be 30.

2. (a) The values $c = \pm 1$ and $c = 1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function $x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.



(b) $\nabla f = \langle 2x, 1 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. So $2x = 2\lambda x \Rightarrow$ either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then $y = \frac{1}{2}$ and so $x = \pm \frac{\sqrt{3}}{2}$ (from the constraint). If $x = 0$, then $y = \pm 1$. Therefore f has possible extreme values at the points $(0, \pm 1)$ and $\left(\pm \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. We calculate $f\left(\pm \frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{5}{4}$ (the maximum value), $f(0, 1) = 1$, and $f(0, -1) = -1$ (the minimum value). These are our answers from (a).

3. $f(x, y) = x^2 - y^2$, $g(x, y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2x, -2y \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2x = 2\lambda x$ implies $x = 0$ or $\lambda = 1$. If $x = 0$, then $x^2 + y^2 = 1$ implies $y = \pm 1$ and if $\lambda = 1$, then $-2y = 2\lambda y$ implies $y = 0$ and thus $x = \pm 1$. Thus the possible points for the extreme values of f are $(\pm 1, 0)$, $(0, \pm 1)$. But $f(\pm 1, 0) = 1$ while $f(0, \pm 1) = -1$ so the maximum value of f on $x^2 + y^2 = 1$ is $f(\pm 1, 0) = 1$ and the minimum value is $f(0, \pm 1) = -1$.

4. $f(x, y) = 4x + 6y$, $g(x, y) = x^2 + y^2 = 13 \Rightarrow \nabla f = \langle 4, 6 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply $x = \dfrac{2}{\lambda}$ and $y = \dfrac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\dfrac{2}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 \Rightarrow 13 = \dfrac{13}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(2, 3)$, $(-2, -3)$. We compute $f(2, 3) = 26$ and $f(-2, -3) = -26$, so the maximum value of f on $x^2 + y^2 = 13$ is $f(2, 3) = 26$ and the minimum value is $f(-2, -3) = -26$.

5. $f(x, y) = x^2 y$, $g(x, y) = x^2 + 2y^2 = 6 \Rightarrow \nabla f = \langle 2xy, x^2 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 4\lambda y \rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $\lambda = y$. If $x = 0$, then $x^2 = 4\lambda y$ implies $\lambda = 0$ or $y = 0$. However, if $y = 0$ then $g(x, y) = 0$, a contradiction. So $\lambda = 0$ and then $g(x, y) = 6 \Rightarrow y = \pm\sqrt{3}$. If $\lambda = y$, then $x^2 = 4\lambda y$ implies $x^2 = 4y^2$, and so $g(x, y) = 6 \Rightarrow 4y^2 + 2y^2 = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. Thus f has possible extreme values at the points $(0, \pm\sqrt{3})$, $(\pm 2, 1)$, and $(\pm 2, -1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2, 1) = 4$ and the minimum to be $f(\pm 2, -1) = -4$.

6. $f(x,y)=x^2+y^2$, $g(x,y)=x^4+y^4=1$ $\Rightarrow$ $\nabla f=\langle 2x,2y\rangle$, $\lambda\nabla g=\langle 4\lambda x^3,4\lambda y^3\rangle$. Then $x=2\lambda x^3$ implies $x=0$ or $\lambda=\dfrac{1}{2x^2}$. If $x=0$, then $x^4+y^4=1$ implies $y=\pm1$. But $y=2\lambda y^3$ implies $y=0$ so $x=\pm1$ or $\lambda=\dfrac{1}{2y^2}$ and $x^2=y^2$ and $2x^4=1$ so $x=\pm\frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0,\pm1)$, $(\pm1,0)$, $\left(\pm\frac{1}{\sqrt[4]{2}},\pm\frac{1}{\sqrt[4]{2}}\right)$, with the maximum value of f on $x^4+y^4=1$ being $f\left(\pm\frac{1}{\sqrt[4]{2}},\pm\frac{1}{\sqrt[4]{2}}\right)=\frac{2}{\sqrt{2}}=\sqrt{2}$ and the minimum value being $f(0,\pm1)=f(\pm1,0)=1$.

7. $f(x,y,z)=2x+6y+10z$, $g(x,y,z)=x^2+y^2+z^2=35$ $\Rightarrow$ $\nabla f=\langle 2,6,10\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$. Then $2\lambda x=2$, $2\lambda y=6$, $2\lambda z=10$ imply $x=\dfrac{1}{\lambda}$, $y=\dfrac{3}{\lambda}$, and $z=\dfrac{5}{\lambda}$. But

$$35=x^2+y^2+z^2=\left(\frac{1}{\lambda}\right)^2+\left(\frac{3}{\lambda}\right)^2+\left(\frac{5}{\lambda}\right)^2 \;\Rightarrow\; 35=\frac{35}{\lambda^2} \;\Rightarrow\; \lambda=\pm1,$$

so f has possible extreme values at the points $(1,3,5)$, $(-1,-3,-5)$. The maximum value of f on $x^2+y^2+z^2=35$ is $f(1,3,5)=70$, and the minimum is $f(-1,-3,-5)=-70$.

8. $f(x,y,z)=8x-4z$, $g(x,y,z)=x^2+10y^2+z^2=5$ $\Rightarrow$ $\nabla f=\langle 8,0,-4\rangle$, $\lambda\nabla g=\langle 2\lambda x,20\lambda y,2\lambda z\rangle$. Then $2\lambda x=8$, $20\lambda y=0$, $2\lambda z=-4$ imply $x=\dfrac{4}{\lambda}$, $y=0$, and $z=-\dfrac{2}{\lambda}$. But

$$5=x^2+10y^2+z^2=\left(\frac{4}{\lambda}\right)^2+10\,(0)^2+\left(-\frac{2}{\lambda}\right)^2 \;\Rightarrow\; 5=\frac{20}{\lambda^2} \;\Rightarrow\; \lambda=\pm2,$$

so f has possible extreme values at the points $(2,0,-1)$, $(-2,0,1)$. The maximum of f on $x^2+10y^2+z^2=5$ is $f(2,0,-1)=20$, and the minimum is $f(-2,0,1)=-20$.

9. $f(x,y,z)=xyz$, $g(x,y,z)=x^2+2y^2+3z^2=6$ $\Rightarrow$ $\nabla f=\langle yz,xz,xy\rangle$, $\lambda\nabla g=\langle 2\lambda x,4\lambda y,6\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies $\lambda=(yz)/(2x)=(xz)/(4y)=(xy)/(6z)$ or $x^2=2y^2$ and $z^2=\frac{2}{3}y^2$. Thus $x^2+2y^2+3z^2=6$ implies $6y^2=6$ or $y=\pm1$. Then the possible points are $\left(\sqrt{2},\pm1,\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2},\pm1,-\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm1,\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm1,-\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

10. $f(x,y,z)=x^2y^2z^2$, $g(x,y,z)=x^2+y^2+z^2=1$ $\Rightarrow$ $\nabla f=\langle 2xy^2z^2,2yz^2x^2,2zx^2y^2\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies (1) $\lambda=y^2z^2=x^2z^2=x^2y^2$ and $\lambda\neq0$, or (2) $\lambda=0$ and one or two (but not three) of the coordinates are 0. If (1) then $x^2=y^2=z^2=\frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$.

11. $f(x,y,z)=x^2+y^2+z^2$, $g(x,y,z)=x^4+y^4+z^4=1$ $\Rightarrow$ $\nabla f=\langle 2x,2y,2z\rangle$, $\lambda\nabla g=\langle 4\lambda x^3,4\lambda y^3,4\lambda z^3\rangle$.

Case I: If $x\neq0$, $y\neq0$ and $z\neq0$, then $\nabla f=\lambda\nabla g$ implies $\lambda=1/(2x^2)=1/(2y^2)=1/(2z^2)$ or $x^2=y^2=z^2$ and $3x^4=1$ or $x=\pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.

21. $P(L,K)=bL^{\alpha}K^{1-\alpha}$, $g(L,K)=mL+nK=p \;\Rightarrow\; \nabla P=\langle \alpha bL^{\alpha-1}K^{1-\alpha}, (1-\alpha)bL^{\alpha}K^{-\alpha}\rangle$, $\lambda\nabla g=\langle \lambda m,\lambda n\rangle$. Then $\alpha b(K/L)^{1-\alpha}=\lambda m$ and $(1-\alpha)b(L/K)^{\alpha}=\lambda n$ and $mL+nK=p$, so $\alpha b(K/L)^{1-\alpha}/m=(1-\alpha)b(L/K)^{\alpha}/n$ or $n\alpha/[m(1-\alpha)]=(L/K)^{\alpha}(L/K)^{1-\alpha}$ or $L=Kn\alpha/[m(1-\alpha)]$. Substituting into $mL+nK=p$ gives $K=(1-\alpha)p/n$ and $L=\alpha p/m$ for the maximum production.

22. $C(L,K)=mL+nK$, $g(L,K)=bL^{\alpha}K^{1-\alpha}=Q \;\Rightarrow\; \nabla C=\langle m,n\rangle$, $\lambda\nabla g=\langle \lambda\alpha bL^{\alpha-1}K^{1-\alpha}, \lambda(1-\alpha)bL^{\alpha}K^{-\alpha}\rangle$. Then $\dfrac{m}{\alpha b}\left(\dfrac{L}{K}\right)^{1-\alpha}=\dfrac{n}{(1-\alpha)b}\left(\dfrac{K}{L}\right)^{\alpha}$ and $bL^{\alpha}K^{1-\alpha}=Q$ $\Rightarrow\; \dfrac{n\alpha}{m(1-\alpha)}=\left(\dfrac{L}{K}\right)^{1-\alpha}\left(\dfrac{L}{K}\right)^{\alpha} \;\Rightarrow\; L=\dfrac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\dfrac{Kn\alpha}{m(1-\alpha)}\right]^{\alpha}K^{1-\alpha}=Q$. Hence $K=\dfrac{Q}{b(n\alpha/[m(1-\alpha)])^{\alpha}}=\dfrac{Qm^{\alpha}(1-\alpha)^{\alpha}}{bn^{\alpha}\alpha^{\alpha}}$ and $L=\dfrac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}}=\dfrac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$ minimizes cost.

23. Let the sides of the rectangle be x and y. Then $f(x,y)=xy$, $g(x,y)=2x+2y=p \;\Rightarrow\; \nabla f(x,y)=\langle y,x\rangle$, $\lambda\nabla g=\langle 2\lambda,2\lambda\rangle$. Then $\lambda=\frac{1}{2}y=\frac{1}{2}x$ implies $x=y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

24. Let $f(x,y,z)=s(s-x)(s-y)(s-z)$, $g(x,y,z)=x+y+z$. Then $\nabla f=\langle -s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y)\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$. Thus (1) $(s-y)(s-z)=(s-x)(s-z)$ and (2) $(s-x)(s-z)=(s-x)(s-y)$. (1) implies $x=y$ while (2) implies $y=z$, so $x=y=z=p/3$ and the triangle with maximum area is equilateral.

25. Let $f(x,y,z)=d^2=(x-2)^2+(y-1)^2+(z+1)^2$, then we want to minimize f subject to the constraint $g(x,y,z)=x+y-z=1$. $\nabla f=\lambda\nabla g \;\Rightarrow\; \langle 2(x-2),2(y-1),2(z+1)\rangle=\lambda\langle 1,1,-1\rangle$, so $x=(\lambda+4)/2$, $y=(\lambda+2)/2$, $z=-(\lambda+2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+4}{2}+\dfrac{\lambda+2}{2}+\dfrac{\lambda+2}{2}=1 \;\Rightarrow\; 3\lambda+8=2 \;\Rightarrow\; \lambda=-2$, so $x=1$, $y=0$, and $z=0$. This must correspond to a minimum, so the shortest distance is $d=\sqrt{(1-2)^2+(0-1)^2+(0+1)^2}=\sqrt{3}$.

26. Let $f(x,y,z)=d^2=(x-1)^2+(y-2)^2+(z-3)^2$, then we want to minimize f subject to the constraint $g(x,y,z)=x-y+z=4$. $\nabla f=\lambda\nabla g \;\Rightarrow\; \langle 2(x-1),2(y-2),2(z-3)\rangle=\lambda\langle 1,-1,1\rangle$, so $x=(\lambda+2)/2$, $y=(4-\lambda)/2$, $z=(\lambda+6)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+2}{2}-\dfrac{4-\lambda}{2}+\dfrac{\lambda+6}{2}=4 \;\Rightarrow\; \lambda=\frac{4}{3}$, so $x=\frac{5}{3}$, $y=\frac{4}{3}$, and $z=\frac{11}{3}$. This must correspond to a minimum, so the point on the plane closest to the point $(1,2,3)$ is $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$.

27. $f(x,y,z)=x^2+y^2+z^2$, $g(x,y,z)=z^2-xy-1=0 \;\Rightarrow\; \nabla f=\langle 2x,2y,2z\rangle=\lambda\nabla g=\langle -\lambda y,-\lambda x,2\lambda z\rangle$. Then $2z=2\lambda z$ implies $z=0$ or $\lambda=1$. If $z=0$ then $g(x,y,z)=1$ implies $xy=-1$ or $x=-1/y$. Thus $2x=-\lambda y$ and $2y=-\lambda x$ imply $\lambda=2/y^2=2y^2$ or $y=\pm1$, $x=\pm1$. If $\lambda=1$, then $2x=-y$ and $2y=-x$ imply $x=y=0$, so $z=\pm1$. Hence the possible points are $(\pm1,\mp1,0)$, $(0,0,\pm1)$ and the minimum value of f is $f(0,0,\pm1)=1$, so the points closest to the origin are $(0,0,\pm1)$.