Part I

1. Find, in the form $y = f(x)$, the general solution of the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + xy = x + 1$$

2. Solve the differential equation

$$x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cos x \ ,$$

given that the $y = \frac{2}{\pi}$ when $x = \frac{\pi}{2}$.

3. Find in the form $y = f(x)$ the general solution to the differential equation. (First convert it into a separable differential equation using the substitution $y = ux$

$$xy \frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$$

Part II

1. A water tank is in the shape of a cylinder with its axis vertical. The area of the base is A m^2. Initially the tank is empty. Starting at time t=0 water is poured into the tank at a constant rate of w m^{3}s^{-1}, and leaks out through a small hole in the base at a rate of kx m^{3}s^{-1}, where k is a constant and x m is the depth of the water in the tank. Show that

$$A\frac{\mathrm{d}x}{\mathrm{d}t} = w - kx\ .$$

Let $A = 1$ in this expression and solve it for x as a function of t.

Show that, however long the process continues, the depth of water never exceeds $\frac{w}{k}$ m, and that if the time taken to reach half this depth is T seconds then $kT = \ln 2$.

2. The rate at which a cup of tea loses temperature T is proportional to the amount by which it exceeds the surrounding temperature T_0 at any given time. If room temperature is 20^0 and the tea cools from 100^o to 80^o in 5 minutes. Find out how long it takes to cool from 80^o to 40^o.