

4. (a) The equilibrium points occur where the vector field is zero, that is, at solutions of

$$\begin{cases} -x = 0 \\ -4x^3 + y = 0. \end{cases}$$

So, $x = y = 0$ is the only equilibrium point.

- (b) The Jacobian matrix of this system is

$$\begin{pmatrix} -1 & 0 \\ -12x^2 & 1 \end{pmatrix},$$

which at $(0, 0)$ is equal to

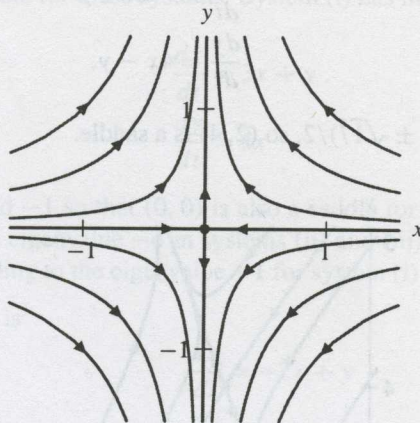
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So the linearized system at $(0, 0)$ is

$$\begin{aligned} \frac{dx}{dt} &= -x \\ \frac{dy}{dt} &= y \end{aligned}$$

(we could also see this by “dropping the higher order terms”).

- (c) The eigenvalues of the linearized system at the origin are -1 and 1 , so the origin is a saddle point. The linearized system decouples, so solutions approach the origin along the x -axis and depart away from the origin along the y -axis.



5. (a) Using separation of variables (or simple guessing), we have $x(t) = x_0 e^{-t}$.
 (b) The equation

$$\frac{dy}{dt} = -4x^3 + y$$

5.1 Equilibrium Point Analysis

is a first-order, linear equation. We write the equation as

$$\frac{dy}{dt} - y = -4x^3.$$

Therefore, the integrating factor is e^{-t} . Multiplying both sides of the equation by e^{-t} yields

$$\left(\frac{dy}{dt} - y\right)e^{-t} = -4x^3e^{-t}.$$

Note that the left-hand side is just the derivative of ye^{-t} , and the right-hand side is $-4x_0^3e^{-3t}e^{-t}$ since $x(t) = x_0e^{-t}$. Therefore we have

$$\frac{d}{dt}ye^{-t} = -4x_0^3e^{-3t}e^{-t} = -4x_0^3e^{-4t}.$$

After integrating and simplifying, we have

$$y(t) = x_0^3e^{-3t} + (y_0 - x_0^3)e^t.$$

(c) The general solution of the system is

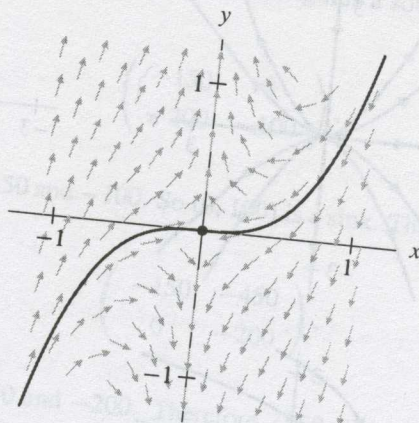
$$x(t) = x_0e^{-t}$$

$$y(t) = x_0^3e^{-3t} + (y_0 - x_0^3)e^t.$$

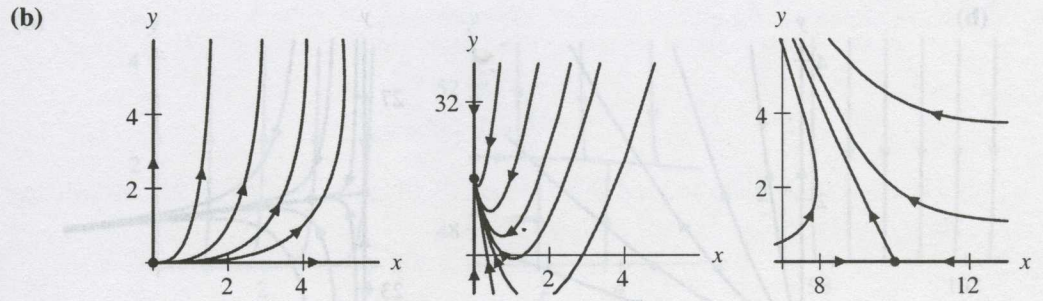
(d) For all solutions, $x(t) \rightarrow 0$ as $t \rightarrow \infty$. For a solution to tend to the origin as $t \rightarrow \infty$, we must have $y(t) \rightarrow 0$, and this can happen only if $y_0 - x_0^3 = 0$.

(e) Since $x = x_0e^{-t}$, we see that a solution will tend toward the origin as $t \rightarrow -\infty$ only if $x_0 = 0$. In that case, $y(t) = y_0e^t$, and $y(t) \rightarrow 0$ as $t \rightarrow -\infty$.

(f)



(g) Solutions tend away from the origin along the y -axis in both systems. In the nonlinear system, solutions approach the origin along the curve $y = x^3$ which is tangent to the x -axis. For the linearized system, solutions tend to the origin along the x -axis. Near the origin, the phase portraits are almost the same.



9. (a) The equilibrium points are $(0, 0)$, $(0, 25)$, $(100, 0)$ and $(75, 12.5)$. We classify these equilibrium points by computing the Jacobian matrix, which is

$$\begin{pmatrix} 100 - 2x - 2y & -2x \\ -y & 150 - x - 12y \end{pmatrix},$$

and evaluating it at each of the equilibrium points. At $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 100 & 0 \\ 0 & 150 \end{pmatrix},$$

and the eigenvalues are 100 and 150. So this point is a source. At $(0, 25)$, the Jacobian matrix is

$$\begin{pmatrix} 50 & 0 \\ -25 & -150 \end{pmatrix},$$

and the eigenvalues are 50 and -150 . Hence, this point is a saddle. At $(100, 0)$, the Jacobian matrix is

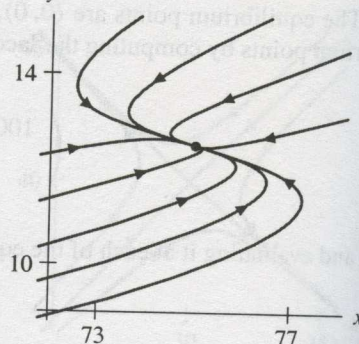
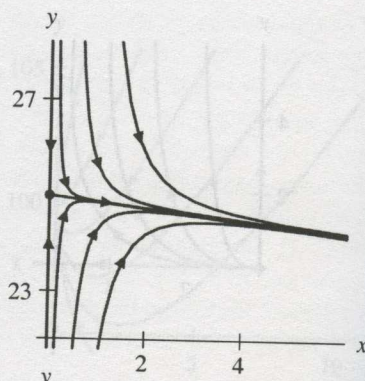
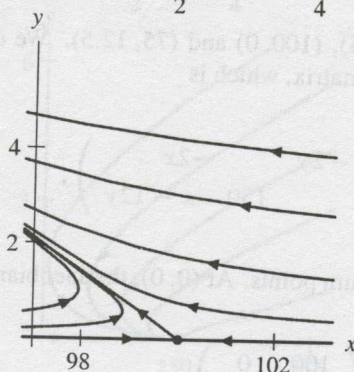
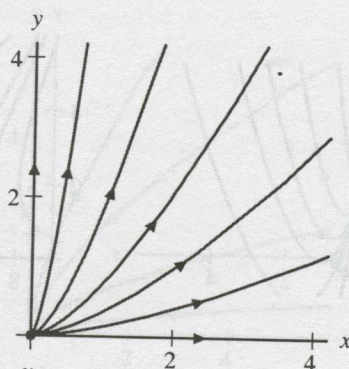
$$\begin{pmatrix} -100 & -200 \\ 0 & 50 \end{pmatrix},$$

and the eigenvalues are -100 and 50. Therefore, this point is a saddle. Finally, at $(75, 12.5)$, the Jacobian matrix is

$$\begin{pmatrix} -75 & -150 \\ -12.5 & -75 \end{pmatrix},$$

and the eigenvalues are approximately -32 and -118 . So this point is a sink.

(b)



10. (a) The equilibrium points in the first quadrant are $(0, 0)$, $(0, 50)$ and $(100, 0)$. To classify these equilibrium points, we compute the Jacobian matrix, which is

$$\begin{pmatrix} -2x - y + 100 & -x \\ -2xy & -x^2 - 3y^2 + 2500 \end{pmatrix},$$

and evaluate it at each point. At $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 100 & 0 \\ 0 & 2500 \end{pmatrix},$$

which has eigenvalues 100 and 2500. So $(0, 0)$ is a source. At $(0, 50)$, the Jacobian matrix is

$$\begin{pmatrix} 50 & 0 \\ 0 & -5000 \end{pmatrix},$$

which has eigenvalues -10 and -5000 . Hence, $(0, 50)$ is a saddle. At $(100, 0)$, the Jacobian matrix is

$$\begin{pmatrix} -100 & -100 \\ 0 & 900 \end{pmatrix},$$

which has eigenvalues -40 and -7500 . Thus, $(100, 0)$ is a sink.

