Non-Linear Dynamics Homework Solutions

Week 4: Strogatz Portion

February 3, 2009

6.1.3 Find fixed points and sketch the nullclines, vector field and a plausible phase portrait of the system

$$\begin{aligned}\dot{x} &= x(x-y)\\ \dot{y} &= y(2x-y)\end{aligned}$$

We get our x-nullclines by setting $\dot{x}=0$ and solving the equation for x and y. We find these to be $x=0$ and $y=x$. Our $\dot{y}=0$ nullclines are going to be given by the lines $y=0$ and $y=2x$. To find our fixed points, we look for where y nullclines intersect with x nullclines, since at those intersections $\dot{x}=\dot{y}=0$. From this it follows that the only fixed point for this system is at the origin. See Figure 1.

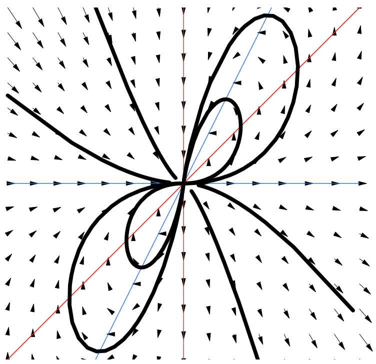

Figure 1: Vector Field with Phase Portrait and Nullclines: The *blue* lines are y-nullclines and the *red* are x-nullclines

6.1.7 *Nullclines vs. Stable Manifolds* Note that it would be impossible for the non-linear curve drawn in Figure 6.1.3 (in Strogatz) to be a stable manifold, since the vector lines aren't even pointing along it there. Solution follow vector lines, but not necessarily nullclines, and in this case there must exist paths crossing the nullclines so the nullclines themselves cannot be paths in the phase portrait. See Figure 2 for the plot.

6.3.6 We find and classify fixed points of the following system

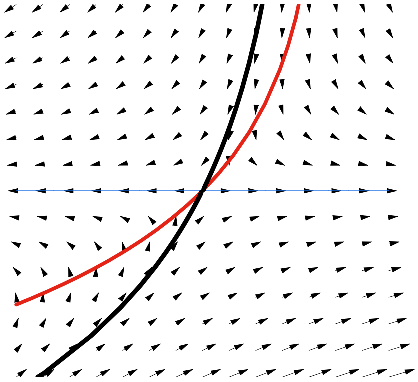

Figure 2: Stable Manifolds in *black* and x-nullcline in *red*

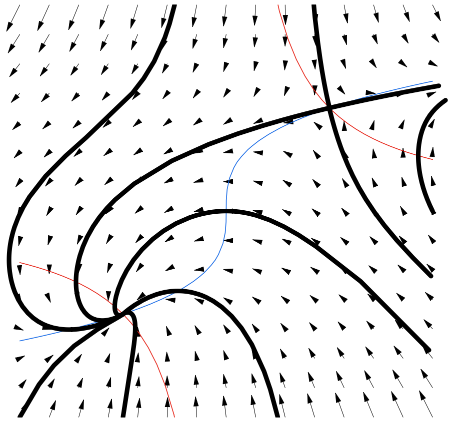

Figure 3: Nullclines, Vector Field, and Phase Portrait

$$\begin{aligned}\dot{x} &= xy - 1\\ \dot{y} &= x - y^3\end{aligned}$$

Setting things equal to zero, we find that the x-nullcline is $y = 1/x$ and our y-nullcline is $y = x^{1/3}$. Intersections occur at points $(1, 1)$ and $(-1, -1)$. To use linear analysis at these points we compute the Jacobian and evaluate at our fixed points giving us the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$$

for the fixed points $(1, 1)$ and $(-1, -1)$ respectively. We calculate the traces and determinants to be $T = -2$ and $D = -5$ for $(1, 1)$ and for $(-1, -1)$ we have $T = -4$ and $D = 4$. This means that the former fixed point is a saddle and the latter is stable and somewhere in between a spiral and a sink. See Figure 3 for a plot of the vector field with nullclines and phase portrait.

6.3.10 *Inconclusive Linearization of Fixed Points* We investigate the system given by

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= x^2 - y \end{aligned}$$

a) To see that the linearization predicts that the origin is a non-isolated fixed point we compute the linearization matrix obtained from the Jacobian of the system evaluated at the origin. We find

$$A = \begin{pmatrix} y & x \\ 2x & -1 \end{pmatrix}_O = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that there is a whole line of zero solutions predicted by this linearization, specifically all of those on the line parameterized by the vector equation

$$\vec{L}(a) = a \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

since $A\vec{L}(a) = 0$ for all $a \in \mathbf{R}$.

b) We show that the origin is indeed an isolated fixed point.

Note that for $\dot{x} = 0$ to be true, one of x or y equals 0. If this is the case, for $\dot{y} = 0$ to be true, the other of x and y must then be zero. Thus the origin is the only fixed point of the system.

c) This is accomplished by finding nullclines, plotting them, figuring out what the flow is like along their various regions, and then filling in the rest of the field as it makes sense to. Then draw phase lines along the flow of the field. See Figure 4 for a computer generated plot of vector field, nullclines and phase portrait.

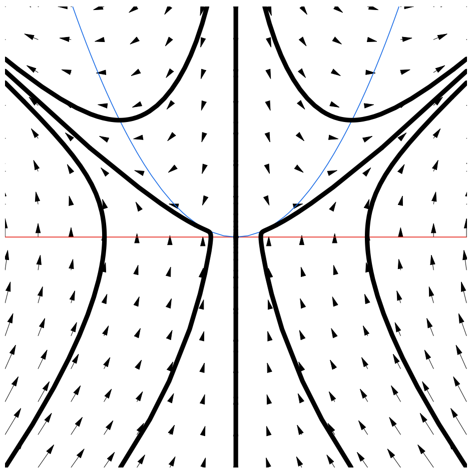

Figure 4: Nullclines, Vector Field, and Phase Portrait

6.3.11 We study the system in radial coordinates given by the equations

$$\dot{r} = -r \tag{1}$$

$$\dot{\theta} = \frac{1}{\ln r}. \tag{2}$$

a) We find $r(t)$ and $\theta(t)$ explicitly.

It's easy to see that $r(t) = r_0 e^{-t}$ is a solution for r. To find a solution for θ, we substitute our equation for r into that of $\dot{\theta}$. This gives us

$$\dot{\theta} = \frac{1}{a-t},$$

where $a = \ln r_0$. We separate variables and then use the substitution $u = a - t$ to solve the resulting integral. Doing this we find that $\theta(t) = -\ln|\ln x_0 - t| + c$ where c is some constant which, using our initial condition we find to be $-\ln|\ln x_0 - t| + \theta_0$ giving us the equation

$$\theta(t) = \ln\left|\frac{\ln x_0}{\ln(x_0) - t}\right| + \theta_0.$$

b) We show that $r(t) \to 0$ and $|\theta(t)| \to \infty$ as $t \to \infty$, implying that the center is a stable spiral for the non-linear system.

This is easy to verify for $r(t)$. As $t \to \infty$, $e^{-t} \to 0$, as we well know. As $t \to \infty$ the term $(\ln x_0)/(\ln(x_0) - t) \to 0$ and $\ln x \to -\infty$ as $x \to 0$ so $|\theta(t)| \to 0$ as $t \to \infty$.

The remainder of the problems of this assignment are from Blanchard and Devaney Chapter 5, and solutions can be found in a separate document on the Math Methods home page.