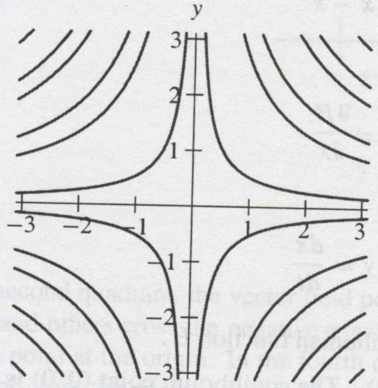
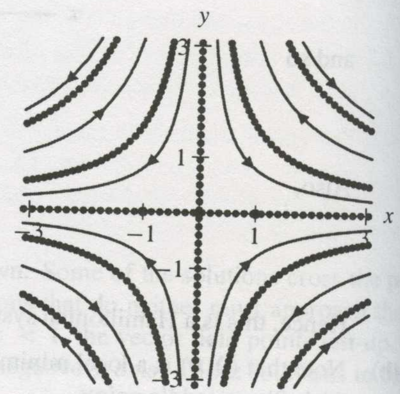
(b) Note that the level sets of H are the same curves as those of the level sets of xy.



(c) Note that there are many curves of equilibrium points for this system: besides the origin, whenever $xy = n\pi + \pi/2$, the vector field vanishes.



3. **(a)** If $H(x, y) = x \cos y + y^2$, then

$$\frac{\partial H}{\partial x} = \cos y$$

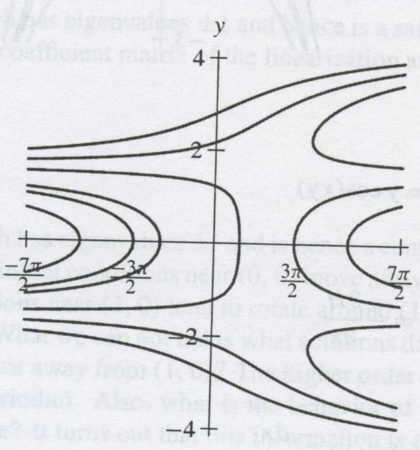
and so

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

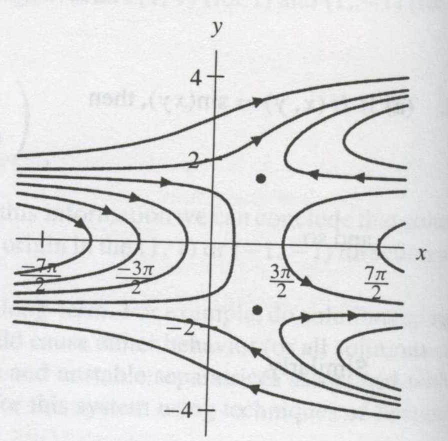
Similarly,

$$\frac{\partial H}{\partial y} = -x \sin y + 2y = \frac{dx}{dt}.$$

(b)



(c) The equilibrium points occur at points of the form $((1-4n)\pi, (2n-\frac{1}{2})\pi)$ and $((1+4n)\pi, (2n+\frac{1}{2})\pi)$ where n is an integer.



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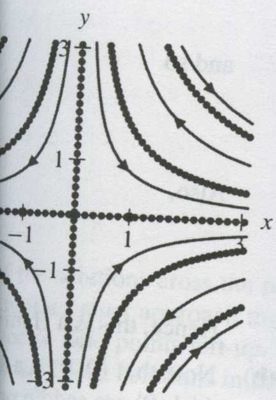
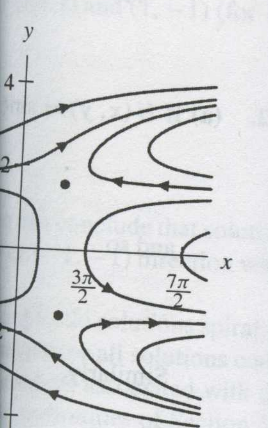


m points occur at points
$-4n)\pi$, $(2n - \frac{1}{2})\pi)$ and
$n + \frac{1}{2})\pi)$ where n is an



4. **(a)** The Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -(g/l)\cos\theta & 0 \end{pmatrix}.$$

At (0, 0), the linearization is

$$\begin{pmatrix} 0 & 1 \\ -g/l & 0 \end{pmatrix}.$$

(b) Note that the equation does not depend on m. Using $g = 9.8$, the eigenvalues for the linearization are $\pm i\sqrt{9.8/l}$ and the period of the solutions is $2\pi/\sqrt{9.8/l}$. Hence we need

$$2\pi/\sqrt{9.8/l} = 1$$

or $l = 9.8/4\pi^2$.

5. A large amplitude swing will take θ near $\pm\pi$, $v = 0$, the equilibrium point corresponding to the pendulum being balanced straight up. Near equilibrium points the vector field is very short, so solutions move very slowly. A solution passing close to $\pm\pi$, $v = 0$ must move slowly and hence, take a long time to make one complete swing. Hence, very high swings have long period. We must also be careful not to let the pendulum "swing over the top".

6. Large amplitude oscillations of an ideal pendulum have much longer period than small amplitude oscillations because they come close to the saddle points. Hence, small amplitude swings make the clock fast.

7. **(a)** The linearization at the origin is

$$\frac{d\theta}{dt} = v$$

$$\frac{dv}{dt} = -\frac{g}{l}\theta.$$

The eigenvalues of this system are $\pm i\sqrt{g/l}$, so the natural period is $2\pi/(\sqrt{g/l})$, which can also be written as $2\pi\sqrt{l}/\sqrt{g}$. Doubling the arm length corresponds to replacing l with $2l$, but the computations above stay the same. The natural period for arm length $2l$ is $2\pi\sqrt{2l}/\sqrt{g}$. Doubling the arm length multiplies the natural period by $\sqrt{2}$.

(b) Compute

$$\frac{d(2\pi\sqrt{l}/\sqrt{g})}{dl} = \frac{\pi}{\sqrt{gl}}.$$

8. Let G be the gravitational constant on the moon. Note that $G < g = 9.8$. The period of the linearization of the ideal pendulum on the moon is $2\pi/\sqrt{G/l}$. Since $G < g$, we have

$$2\pi/\sqrt{G/l} > 2\pi/\sqrt{G/l}.$$

Since the period of the pendulum is now longer, the clock runs more slowly.

9. We know that the equilibrium points of a Hamiltonian system cannot be sources or sinks. Phase portrait (b) has a spiral source, so it is not Hamiltonian. Phase portrait (c) has a sink and a source, so it is not Hamiltonian. Phase portraits (a) and (d) might come from Hamiltonian systems. (Try to imagine a function which has the solution curves as level sets.)

10. First note that

$$\frac{\partial(\sin x \cos y)}{\partial x} = \cos x \cos y = -\frac{\partial(2x - \cos x \sin y)}{\partial y}.$$

Hence, the system is Hamiltonian. Integrating dx/dt with respect to y yields

$$H(x, y) = \sin x \sin y + c(x).$$

If we differentiate $H(x, y)$ with respect to x, we get

$$\cos x \sin y + c'(x),$$

which we want to be the negative of $dy/dt = 2x - \cos x \sin y$. Hence $c'(x) = -2x$, and we pick t antiderivative $c(x) = -x^2$. A Hamiltonian function is

$$H(x, y) = -x^2 + \sin x \sin y.$$

11. First note that

$$\frac{\partial(x - 3y^2)}{\partial x} = 1 = -\frac{\partial(-y)}{\partial y}.$$

Hence, the system is Hamiltonian. Integrating dx/dt with respect to y yields

$$H(x, y) = xy - y^3 + c(x).$$

If we differentiate $H(x, y)$ with respect to x, we get

$$y + c'(x),$$

which we want to be the negative of $dy/dt = -y$. Hence $c'(x) = 0$, and we pick the antideriva $c(x) = 0$. A Hamiltonian function is

$$H(x, y) = xy - y^3.$$

12. First we check to see if the partial derivative with respect to x of the first component of the ve field is the negative of the partial derivative with respect to y of the second component. We have

$$\frac{\partial 1}{\partial x} = 0$$

while

$$-\frac{\partial y}{\partial y} = -1.$$

Since these are not equal, the system is not Hamiltonian.

13. First we check to see if the partial derivative with respect to x of the first component of the ve field is the negative of the partial derivative with respect to y of the second component. We have

$$\frac{\partial(x \cos y)}{\partial x} = \cos y$$

while

$$-\frac{\partial(-y \cos x)}{\partial y} = \cos x.$$

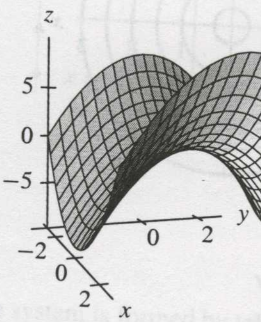
Since these two are not equal, the system is not Hamiltonian.

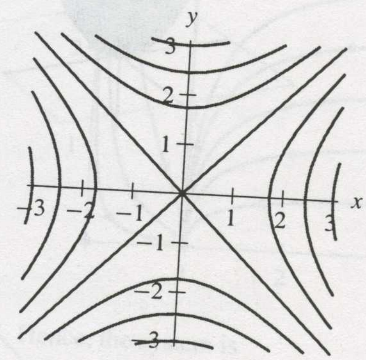
13. **(a)** We have $\nabla G(x, y) = (2x, -2y)$, so

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = -2y.$$

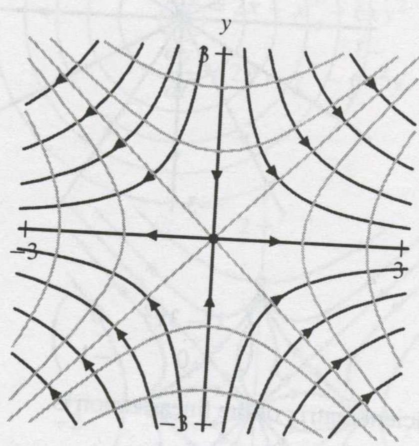
(b) The system is linear and has eigenvalues 2 and -2. Hence the origin is a saddle.

(c) The graph of G is a saddle surface turning up in the x-direction and down in the y-directio





(d) The line of eigenvectors for eigenvalue 2 is the x-axis, the line of eigenvectors for eigenvalu -2 is the y-axis (see Chapter 3).



Phase portrait shown with level sets of G in gray.

14. **(a)** The gradient $\nabla G(x, y) = (2x, 2y)$ so

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = 2y.$$