Non-Linear Dynamics Homework Solutions

Week 5: Strogatz Portion

February 23, 2009

6.5.2 Consider the system $\ddot{x} = x - x^2$.

a) Find and classify all equilibrium points. First we write this as a two dimensional system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^2 \end{aligned}$$

We now see that if $\dot{y} = 0$ then $x = 0$ or $x = 1$. In either case $y = 0$ must be true for the point to remain fixed. We now compute the Jacobian of the system.

$$J = \begin{pmatrix} 0 & 1 \\ 1 - 2x & 0 \end{pmatrix}$$

Analyzing at our fixed points we find that for (0,0) we have $T = 0$ and $D = -1$ making this fixed point a saddle. For $(1, 0)$, $T = 0$ and $D = 1$ implying that the fixed point is a linear center.

Note that this system has the same form as the one discussed on pages 159 and 160 of Strogatz. Using that treatment, we find that

$$E(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$$

is a conserved quantity for this system, implying that our fixed point $(1, 0)$ is a non-linear center.

Evaluated at the origin, this goes to 0. Since trajectories follow contour lines for conservative systems, we shall determine where $E(x, y) = 0$ and know that any trajectories must lie entirely within the subset o f $\mathbf{R}^2$ for which E evaluates to zero. We set $E(x, y) = 0$ and solve for y. Doing this we find that the trajectories leading to the origin lie along the curve

$$y = \pm x\sqrt{1 - \frac{2x}{3}}.$$

Figure 1 shows that we have a homoclinic orbit and that it seperates closed orbits from the rest of the phase plane.

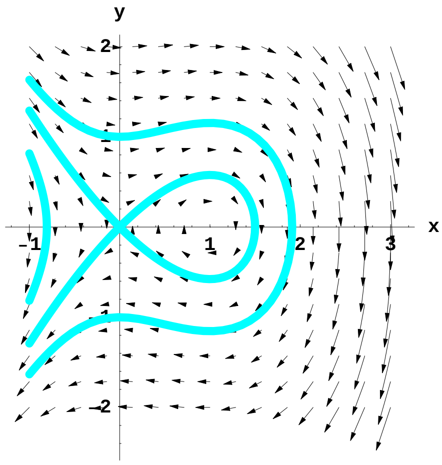

Figure 1: Vector Field and Contour Lines of $E(x, y)$

6.5.7 (General Relativity and Planetary Orbits) We study the relativistic equation for the orbit of a planet around the sun

$$\frac{d^2u}{d\theta^2} + u = \alpha + \mathcal{E}u^2.$$

Here $u = 1/r$ and r, θ are the polar coordinates of the planet in its plane of motion, $\alpha > 0$ and $\mathcal{E} > 0$ but small.

a) Rewrite the system in the (u, v) phase plane, where $v = du/d\theta$.

We find that

$$\begin{aligned} \frac{du}{d\theta} &= v \\ \frac{dv}{d\theta} &= \alpha + \mathcal{E}u^2 - u \end{aligned}$$

b) Find all of the equilibrium points of the system.

We find equalibrium points by solving for v and u in the equations from part b) given that $du/d\theta = dv/d\theta = 0$. Doing this we find that the our equilibrium points are

$$\begin{aligned} \vec{u}_1^* &= \left(\frac{1 + \sqrt{1 - 4\mathcal{E}\alpha}}{2\mathcal{E}}, 0\right) \\ \vec{u}_2^* &= \left(\frac{1 - \sqrt{1 - 4\mathcal{E}\alpha}}{2\mathcal{E}}, 0\right) \end{aligned}$$

c) Show that one of the equilibria is a center in the (u, v) phase plane, according to the linearization about that point. Is it a *nonlinear* center?

We calculate the Jacobian

$$J = \begin{pmatrix} 0 & 1 \\ 2\mathcal{E}u - 1 & 0 \end{pmatrix}$$

When evaluated at our fixed points, we get that $T = 0$ for both and $D = \mp\sqrt{1 - 4\mathcal{E}\alpha} \approx \pm 1$, with the $-$ for $\vec{u}_1^*$ and the $+$ for $\vec{u}_2^*$, where the approximation holds because of the fact that $\mathcal{E}$ is small. This shows that $\vec{u}_1^*$ is a linear center. This is a nonlinear center as well, since the system is conservative, by Theorem 6.5.1 from Strogatz. (Note that we can rewrite this system in the same form as Equation (1) on page 159 of Strogatz. From this it follows that the energy of the system can be expressed in the form given on the next page. Since this system has an energy function, it is conservative.)

d) Show that the equilibrium point found in (c) corresponds to a circular planetary orbit.

The fixed point in question is $u \approx 1/\mathcal{E}$, by our earlier mentioned approximation. Then since $u = 1/r$, this trajectory corresponds to a constant r value of $\mathcal{E}$. Also since $v = 0$ for our fixed point, we have that $0 = du/d\theta = (du/dr)(dr/d\theta)$. Since $du/dr = -r^{-2} = -1/\mathcal{E}^2$ for the fixed point in question, which is not zero, so $dr/d\theta = 0$. Thus we have that the fixed point is a circular path, since the path has constant r values.

6.5.14 (*Glider*) Let v be the glider's speed, θ be the angle it makes to the horizontal. We model the dynamics of these quantities by the equations

$$\begin{aligned} \dot{v} &= -\sin\theta - Dv^2 \\ v\dot{\theta} &= -\cos\theta + v^2 \end{aligned}$$

where the trigonometric terms represent the effects of gravity and the v^2 terms represent the effects of drag and lift.

a) Suppose that there is no drag so that $D = 0$. Show that $E(v, \theta) = v^3 - 3v\cos\theta$ is a conserved quantity. Sketch a phase portrait of the system under these assumtions and interprest the results. What does the flight path of the glider look like?

To show that the quantity is conserved we differentiate with respect to t. If the result is 0, then the quantity is conserved. Aplying the product and chain rules we get

$$\frac{dE}{dt} = 3v^2\dot{v} - 3(\dot{v}\cos\theta - \sin\theta v\dot{\theta})$$

From here we replace $\dot{v}$ and $v\dot{\theta}$ with the equaivalent expressions in terms of v and θ. From that point it is staightforward to verify that all of the terms cancel, giving us $\dot{E} = 0$, which implies that the quantity is indeed conserved, as desired.

Next, we'll find fixed points. Now, $\dot{v} = 0 \Longrightarrow \theta = k\pi$ for some $k \in \mathbf{Z}$, since $D = 0$. For $\dot{\theta} = 0$ to be true, $\cos\theta = v^2$ must be. Thus k must be even (Otherwise we wouldn't be able to take the square root in order to solve for v). Then $v = \pm 1$ so our fixed points are $(\pm 1, 0)$ (if we restrict ourselves to θ values between $\pm\pi$). This means that the glider's motion is stable when moving at velocity ± 1 paralell to the horizontal. Trajectories inside the region bounded by a circle in Figure 3 will be orbits, and correspond to oscilation of angle and velocity. Those outside correspond to loop-dee-loops since θ is unbounded in these regions.

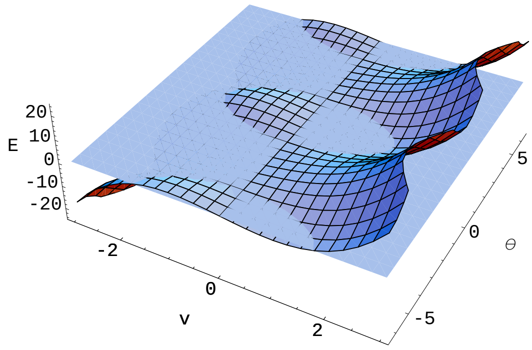

Figure 2: 3-D Energy Plot Showing $E = 0$ intersections

Next we find the Jacobian of the system.

$$J = \begin{pmatrix} 0 & -\cos\theta \\ 1 + \frac{\cos\theta}{v^2} & \frac{\sin\theta}{v} \end{pmatrix}$$

Evaluating this matrix at our fixed points $(\pm 1, 0)$, we get

$$A = \begin{pmatrix} 0 & -1 \\ \pm 2 & 0 \end{pmatrix}$$

This gives us $T = 0$ and $D = 2$ for both fixed points, implying that both are linear centers. Since this system is conservative, it also implies that both are nonlinear centers.

One of the next things we can do which is really helpful is look at contour lines of our energy function, since our trajectories will lie along those. It turns out that $E = 0$ is a good one of these to look at since it seperates different system behaviors. We plot this contour line and two others. This gives us the information we need to start plotting. First we show a plot of the energy of the system and it's intersection with the $E = 0$ plane, followed by the phase portrait on top of a vector field plot.

b) In case of positive drag ($D > 0$), we find that the Jacobian matrix becomes

$$J = \begin{pmatrix} -2Dv & -\cos\theta \\ 1 + \frac{\cos\theta}{v^2} & \frac{\sin\theta}{v} \end{pmatrix}$$

We now also have different fixed points. Solving for the system when $\dot{\theta}$ and $\dot{v}$ are 0, we find that

$$\begin{aligned} v &= \pm\sqrt{\cos(\arctan -D)} \\ \theta &= \arctan -D \end{aligned}$$

gives us our fixed points.

When we plot these in Figures 4 and 5, we see that both the stable velocity and angle values decrease as damping increases. This makes intuitive sense. If the glider experiences

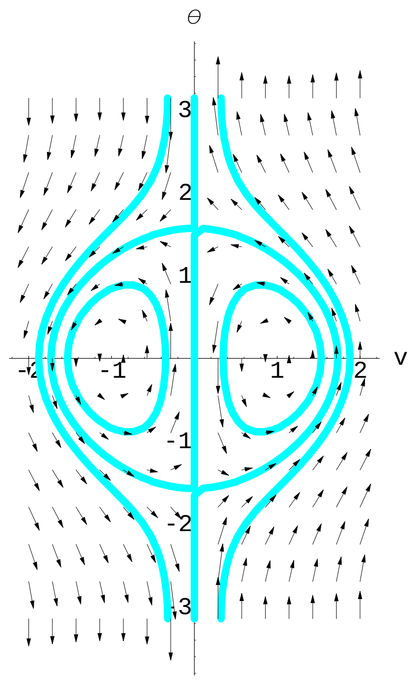

Figure 3: Vector Field and Contour Lines

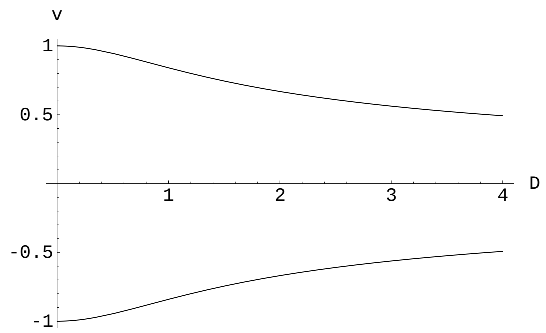

Figure 4: Plot of the v-value of the system's fixed points vs. D.

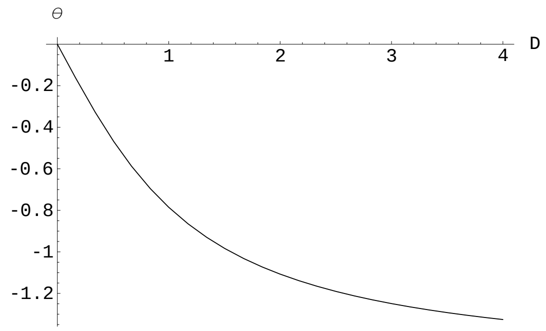

Figure 5: Plot of the θ-value of the system's fixed points vs. D.

resistence, it's going to loose energy and thus needs to decline in altitude slightly (ie, have a negative θ value) so that the speeding up from conversion of potential energy into kinetic balances out the drag force which is absorbing the glider's kinetic energy. Also, the stable velocity becomes lower, also making good sense with the presence of higher drag.

Since we are loosing energy to drag, as D increases, we would expect Figure 2 to change in such a manner that the peak (fixed point) at (-1,0) will shift, as discussed earlier, and will also become unstable, since the trajectories will want to fall down the hill of energy, since energy is being absorbed by the air. The fixed point (1,0) will shift and become a stable sink, since if it's loosing energy its going to spiral downward. Note that this argument is not too rigorous, but if we were motivated enough we should analytically be able to show the same thing by plugging in our new fixed points in terms of D into our Jabobian, giving us the trace and determinant as functions of D. It's a lot of work though, and it's probably pretty clear without doing it.

6.5.19 (Rabbits vs. Foxes) Study the *Lotka-Voltera predator-prey model* given by the system of equations

$$\begin{aligned} \dot{R} &= aR - bRF \\ \dot{F} &= -cF + dRF \end{aligned}$$

.

a) Discuss the biological interprestations of the various parameters and comment on unrealistic ussumptions.

Here a is a parameter which is indicates how quickly rabits bread. The value of b gives a measure of how easy it is for a fox to catch a rabbit. The value of c corresponds to how intense competition is between foxes, while d gives a measure of the ease and overall benefit of catching a rabit for the fox population.

What is unrealistic about this model is that if there are no foxes the rabits grow exponentially forever, a clear impossibility. Also, it assumes that the foxes have no other source of food, for they would die if $R = 0$ was true.

b) Show that the model can be recast in dimensionless form as

$$x' = x(1-y) \quad (1)$$
$$y' = \mu y(x-1) \quad (2)$$

To solve this problem we factor a term of aR from $\dot{R}$ and a term of cF from $\dot{F}$. This makes it clear that we need to set $y = bF/a$ and $x = dR/c$. Doing this gives us the $(1-y)$ and $(x-1)$ terms as we would like them. We are now going to define some $\tau(t)$ such that when we compute $dx/d\tau$ we get what we're looking for. Well, we use the chain rule to write out

$$\frac{dx}{d\tau} = \frac{dx}{dR}\dot{R}\frac{dt}{d\tau} = x(1-y)$$

We have $\dot{R}$ and can find dx/dR easily; plugging these things into the desired equation, we deduce that $dt/d\tau = a^{-1}$ which upon sepeartion of variables and integration gives us the equation $\tau = at$. Thus we have x' in the right form, so we check out y' using the chain rule we get the equation $y' = (c/a)y(x-1)$, so we can set $\mu = c/a$ and we have the desired result. Checking that this is dimensionless is relatively straightforward. We must simply check that x and y and μ are unitless based off of their definitions.

c) Find a conservative quantity in terms of the dimensionless variables.

We will do this by finding dy/dx, which when evaluated at some given point (x, y) will give you the slope of the line tangent to the trajectory at that point. The differential equation this then gives us is useful because we can use it to solve for the relationship between y and x along solution curves.

By the chain rule, $dy/dx = (\mu y(x-1))/(x(1-y))$. Applying seperation of variables and doing a couple of simple integrals, we arrive at the equation

$$yx^{\mu} = Ae^{\mu x + y}.$$

Given some initial state $\vec{x}_0$, we can solve for A, and the equation we came up with involving A, x, and y will describe the trajectory the system takes from that initial point. A must be the same for all points alllong that trajectory, so if we solve for A in terms of x and y then we get the conserved quantity we are looking for. (Each value of that quantity corresponds to one of it's own contours as well as to a trajectory of the system which follows that contour)

Solving for A, we have

$$A(x, y) = \frac{yx^{\mu}}{e^{\mu x + y}}$$

.

d) Show that the model predicts cycles in the populations of both species, for almost all intitial conditions.

Note that along the x and y axes, $A = 0$. Asuming that both x_0 and y_0 are poitive, the value of A will be also. Thus the contour of the conserved quantity which the system will follow given these starting points will nexer cross any of the axes. The index of this fixed point can be shown to be zero. Note that the x-nullclines are $x = 0$ $y = 1$ and the y-nullclines are $y = 0$ and $x = 1$. Since these are the only nullclines we know that if we divide the plane into four regions by the lines that intersect $(1, 1)$, then in those regions,

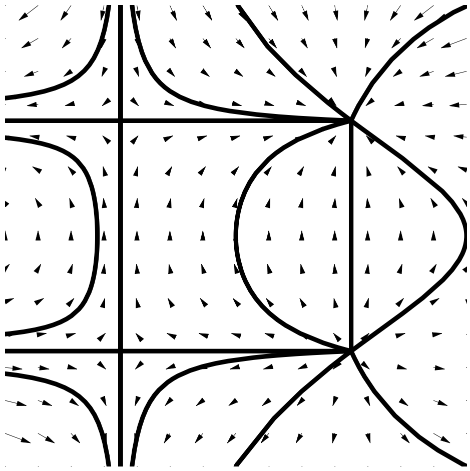

Figure 6: Vector Field of the System from 6.6.1

the angle that the vectors must be making with repect to the x-axis must be inbetween the angles that we get on the nullclines bordering that region. This follows from continuity of the vector field. Consequently, any circle drawn around $(1, 1)$ that is entirely contained within Quadrant I of the x-y plane will have an index of 1. Assuming otherwise leads to an easy contradiction of what was just said before reguarding the regions. Thus, by the system's conservativity, each trajectory lying in the first quadrant must be a closed orbit. A graphical arguement might be the most assuring evidence though. I'll see if I can get one in soon.

6.6.1 Show that the following system is reversible

$$\begin{aligned} \dot{x} &= y(1-x^2) \\ \dot{y} &= 1-y^2 \end{aligned}$$

Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$. Now we have the problem set up as is mentioned on page 164 of Strogatz, and as mentioned there, if we show that $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$ we will have shown that the system is reversible. Well, we can check:

$$f(x, -y) = (-y)(1-x^2) = -(y(1-x^2)) = -f(x, y)$$
$$g(x, -y) = 1-(-y)^2 = 1-(y)^2 = g(x, y)$$

So we have that which was to be demonstrated.

See Figure 6 for a vector plot of the system.

6.6.10 Is the origin a nonlinear center for the following system?

$$\begin{aligned} \dot{x} &= -y-x^2 \\ \dot{y} &= x \end{aligned}$$

We compute the Jacobian matrix

$$J = \begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix}$$

We can see that at the origin we get $T = 0$ and $D = 1$, which certainly makes this a linear center. We'll have that it is a nonlinear center also if we can show that the system is reversible (which it probably is, since this problem is in the Reversible Systems section).

Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$. We show that this system is reversible for x. (This works about the same as for y, we just have to shuffle things around a bit.)

$$g(-x, y) = (-x) = -(x) = -g(x, y)$$
$$f(-x, y) = -y - (-x)^2 = -y - (x)^2 = g(x, y)$$

And so we're done.