

This is an open book test. You may refer to your notes and textbooks.

1. Sketch examples of phase portraits before, at, and after each of the following possible bifurcations:

- (a) An infinite period bifurcation.

- (b) A degenerate Hopf bifurcation.

2. A system is given in polar form below

$$\dot{r} = r(r^2 - \mu r + 1) \quad \text{and} \quad \dot{\theta} = 1$$

For what value of the parameter μ is there a bifurcation? What type of bifurcation is it? Sketch the phase portrait before and after the bifurcation.

3. The following systems represents a model for the autocatalytic chemical reaction that is thought to govern circadian rhythms in the human body

$$\begin{aligned}\dot{x} &= 1 - axy \\ \dot{y} &= axy - \frac{by}{1+y}\end{aligned}$$

x and y are non-dimensionality concentrations of two chemicals and a and b are positive rates.

- (a) Find the fixed point of the system, and determine the conditions on the rate constants a and b so that the fixed point is in the first quadrant of phase space.
- (b) Use the Jacobian to classify the fixed point. In particular show that there is a critical value $a = a_c$ at which there is a bifurcation in stability. Express a_c in terms of b .
- (c) Sketch the nullclines assuming the condition in part (a) applies. (Take $b = 2$ if it makes your life easier.) Construct as complete a trapping region as possible. Most trapping boundaries should be straightforward to find, however the region near the origin along the x axis may cause difficulty. You are not required to find the trapping boundary here but bonus points will be awarded if you do.
- (d) What type of bifurcation do your answers to (b) and (c) suggest occur at $a = a_c$.

4. Consider the following system.

$$\dot{x} = y, \quad \dot{y} = x(1 - z) - ay \quad \dot{z} = -z + x^2$$

- (a) Find and classify the fixed points of this system.
- (b) There is a Hopf bifurcation in this system. Determine the critical value of a_c at which it occurs.
- (c) Estimate the period of the oscillations near the critical point $a = a_c$.