Take home test. You may use your notes and textbooks, but do not consult with others. The test is due on Thursday at 9:00 am.

1. Alice and Bob orbit a distant black hole in stable circular orbits, with Bob at $r = 6M$ and Alice at $r = 12M$.

   (a) What is the distance between their orbits?

   (b) According to a distant observer how fast are each of them going?

   (c) If the light from the the star pulses in intensity and Alice measures a time interval $\Delta\tau_A$ between the pulses what time interval would Bob measure?

2. In Schwarzschild geometry there are two circular orbits for a given angular momentum ℓ provided the angular momentum is bigger than $\sqrt{12}M$. One orbit is stable and one is unstable. In this question you will explore the relationship between

   (a) Suppose the radius of a particular stable circular orbit is $r = aM$. Show that in this case the angular momentum
   $$\ell = \frac{aM}{\sqrt{a-3}}$$

   (b) Hence find an expression for the radius of the unstable circular orbit with the same angular momentum, expressing your answer in terms of a and M.

   (c) Show that these two orbits have equal radial coordinate for $a = 6$, This this is called the innermost stable circular orbit ISCO although it could also be called the outermost unstable circular orbit.

   (d) As a increases the radial coordinate of the unstable orbit decreases. What is the limiting value of the radial coordinate of the unstable orbit as the radius of the stable orbit becomes larger. This value corresponds to the innermost unstable circular orbit.

   (e) Find the period and velocity of a particle in this innermost unstable circular orbit, as measured by a distant observer.

3. A particle moves with velocity v_0 at a large distance from a star of mass M. Its impact parameter is b. The particle passes by the star and is deflected from its path, but misses the star and continues on its way. Its closest approach is given by Schwarzschild radial coordinate $r = R$.

   (a) Show that the energy per unit rest mass, e, is approximately $e = \gamma_0$ and the angular momentum per unit rest mass, ℓ, is approximately $\ell = b\gamma_0 v_0$ where $\gamma_0 = \frac{1}{\sqrt{1-v_0^2}}$.

   (b) In principle R can be determined from v_0 and b using a conservation of energy argument and the fact that at closet approach $dr/d\tau = 0$. Write down, but do not evaluate and equation relating R to b and v_0.

   (c) Find the angular velocity and speed of this particle at the point of closest approach, as measured by a distant observer? Express your answer in terms of R, b and v_0.