One of the important predictions of Einstein's field equations is that the universe is expanding. Each point in space moves away from each other point. If the expansion is homogeneous and isotropic (the same at all places and expanding the same in all directions) this implies that distant galaxies can recede from us at a speed greater than the speed of light as we will show. This may appear to contradict Einstein's second postulate, but in fact there is no contradiction.

1. Uniform expansion means that if the separation d between two galaxies varies in time with the relationship $d = f(t)$, then the separation between any other two galaxies in the universe satisfies $d = kf(t)$, were k is the ratio of the two distances. Show that this implies that at any given moment in time, the speed v of galaxies as they move away from earth is proportional to their distance d away from earth. That is, show $v = Hd$ where H is a proportionality constant that depends on $f(t)$. H is called the Hubble constant. It the same at all points in space, but it may change in time. We currently believe H is increasing.

2. The simplest model for an expanding universe is given by the de Sitter metric
$$ds^2 = -dt^2 + a(t)^2 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) .$$
Show that for this metric $H = \dot{a}/a$.

3. Light from distant galaxies will be red-shifted due to to the expansion of the universe. This is called the cosmological red-shift (which is in addition to any doppler red-shift associated with relative motion of the galaxies). The red-shift z is defined to be $z = \Delta\lambda/\lambda$, where λ is the wave length of the light emitted by the galaxy and $\Delta\lambda$ is the amount by which the wave length increases. The cosmological red-shift depends on the nature of the scale factor $a(t)$, which in turn depends on the density of matter, radiation and vacuum energy of the universe. For a universe expanding at its critical density the distance to the nearest galaxy can be determined from the red-shift by the following formula.
$$d = \frac{2}{H}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

4. Beyond what red shift value z is the recession speed of a the galaxy greater than c? For reference, the largest red-shifts measured are around 6 for distant quasars.

5. What is the largest value of z that is theoretically detectable? What recession speed does this correspond to?

6. Your answer to the above question should have been larger than the speed of light. This would be a problem if an inertial frame of unlimited spatial and temporal extent could be set up. Explain why such a frame cannot be constructed in an expanding universe.

7. Bursts of γ rays from space are detected at the rate of about one per day coming from random locations in the sky. Assuming the bursts are coming from galaxies out to a red-shift of $z = 2$, estimate how long we should way until such a burst occurs in our own galaxy. The average density of galaxies is 0.03 galaxies per Mpc^3. The current value of the Hubble constant is 60 km/s per Mpc. (This is in $c \neq 1$ units. Note 1 Mpc is 1 Mega parsec, which is standard cosmological unit and equals 3.26×10^7 light years. You do not need to convert to light years to solve this problem.)