Quantization of angular momentun L magnitude: 8 May 2007 - E92

\[ E = T + V = \frac{p^2}{2\mu} + \frac{\ell^2}{2\mu r^2} + V(r) \]  

\[ \hat{p}_x = -i\hbar \frac{\partial}{\partial x}, \quad \hat{p}_{x_i} = (-i\hbar) \frac{\partial}{\partial x_i} = -i\hbar \frac{\partial}{\partial x} \quad \text{(momentum operators)} \]

**Spherical Coordinates:**

\[ \hat{p}_r = (-i\hbar) \frac{\partial}{\partial r} = -i\hbar \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r}) \]

\[ \hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}, \quad \hat{L}_r = -i\hbar \frac{\partial}{\partial \phi}, \quad \hat{L} = -i\hbar (\hat{L}_r \hat{L}_z + \hat{L}_z \hat{L}_r) \]

\[ \hat{L}^2 = -i\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \]

(7.20)

Find \( \hat{L}^2 \) in Schrödinger Eqn in Spherical Coords:  

\[ -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{\mu r^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) + V(r) \Psi = E \Psi \]

Separate variables: let \( \Psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \), then Schr. Eqn becomes

\[ \frac{1}{R} \frac{d}{dr} \left( R r^2 \frac{dR}{dr} \right) + \frac{2\mu \hbar^2}{\hbar^2} \left( E - V(r) \right) = \frac{\hbar^2 \ell^2}{2\mu r^2} \]

(7.11) 295

Multiply \[ \frac{\partial^2}{\partial r^2} \] by \( \hbar^2 f(\phi) g(\phi) \) = \( \hbar^2 Y_{\ell m}(\theta, \phi) \) (Spherical harmonics)

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} = \hbar^2 \ell (\ell + 1) Y_{\ell m}(\theta, \phi) \]

(7.21a) 298

So as

\[ \hat{L}^2 Y_{\ell m}(\theta, \phi) = \hbar^2 \ell (\ell + 1) Y_{\ell m}(\theta, \phi) \]

or, since \( \Psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \)

\[ \hat{L}^2 \Psi(r, \theta, \phi) = \hbar^2 \ell (\ell + 1) \Psi(r, \theta, \phi) \]

Therefore, eigenvalues are

\[ \ell = \sqrt{\hbar^2 \ell (\ell + 1)} = \hbar \sqrt{\ell (\ell + 1)} \]
Space quantization of angular momentum $L_z$:

$$\hat{L}_z = -i \hbar \frac{\partial}{\partial \varphi} \Rightarrow \hat{L}_z^2 = -\hbar^2 \frac{\partial^2}{\partial \varphi^2}. $$

1. Find $L_z^2$ in Schrödinger Eqn in spherical coord:

$$-\frac{\hbar^2}{2\mu} \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) - \frac{\hbar^2}{2\mu r^2} \left[ \frac{\sin \Theta}{\sin \Theta d\Phi} (\sin \Theta \frac{\partial}{\partial \Theta}) + \frac{\sin^2 \Phi}{\sin \Theta d\Phi} \right] + V(r) \Psi = E \Psi \tag{7.12}$$

Angular part of the equation approx. to:

$$-m^2 = \frac{1}{g(\Theta)} \frac{d^2 g}{d\phi^2} = -\delta(\Phi) \sin^2 \Theta - \frac{\sin \Theta}{g(\Theta)} \frac{d}{d\Theta} \left[ \sin \Theta \frac{d g(\Theta)}{d \Theta} \right]$$

Solve for $g(\Phi)$:

$$\frac{d^2 g}{d\Phi^2} = -m^2 g \quad \text{What functions have own second deriv?}
\sin, \cos, \exp$$

Try $g = e^{im\Phi}$, then
$$\frac{d g}{d\Phi} = i m e^{im\Phi}$$
$$\frac{d^2 g}{d\Phi^2} = (im)^2 e^{im\Phi} = -m^2 e^{im\Phi} = -m^2 g$$

Singular valued condition on $\Psi$ implies that $g(\Phi + 2\pi) = g(\Phi)$, which requires that $m$ be an integer ($\pm$)

It can be shown that the solution to $[\text{RHS}] = -m^2$ is

$$\Psi(\Theta) = \text{Legendre functions}$$

$$f(\Theta) = \left( \frac{\sin \Theta}{2^l l!} \right)^{1/2} \int_0^{\pi} \left( \cos \Theta - 1 \right)^{l-1/2}$$

Limits on $l$ and $m$ find $f(0)$ and $f(\pi)$. What is required to keep $f(\Theta)$ finite?

$$l = 0, 1, 2, \ldots \quad m = 0, \pm 1, \ldots, \pm l$$

$$Y_{lm}(\Theta, \Phi) = f(\Theta) \Phi(\Phi) = \text{spherical harmonics}$$

Solve for any spherical VC.