Separation of Variables: Laplace's equation \( \nabla^2 V = 0 \) in cylindrical coordinates

Worksheet for fall E&M Problem 3.23 (p.145)

Solve Laplace's eqn by separation of variables in cylindrical coordinates, assuming there is no dependence on \( z \) (this is cylindrical symmetry).

The Laplacian in cylindrical coordinates is eqn (1.82) p.44:

\[ (1) \]

Look for solutions of the form \( V(s, \phi) = S(s) \Phi(\phi) \) \( (2) \)

Multiply by \( s^2 \) and divide by \( V = S \Phi \):

Both terms must be constant, and they must sum to zero, so the two constants are equal and opposite. Choose \(-k^2\) for the \( \Phi \) solution so it returns to its original value in one cycle. Find solutions to (3) and (4)

\[ \frac{s \frac{d}{ds} \left( s \frac{dS}{ds} \right)}{s} = C_1 \quad \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = C_2 = -k^2 \]

(4) Since \( \Phi(\phi + 2 \pi) = \Phi(\phi) \), \( k \) must be an integer.

Show that \( S = s^n \) is a solution to (3): what is the relation between \( n \) and \( k \)?

Show that for \( k \) not 0, \( S = A s^k + B s^{-k} \), and for \( k = 0 \), \( S = D + C \ln s \).

What is \( \Phi \) for the \( k = 0 \) case? Put it all together into a general solution. Then apply it to problem 3.24.