Vector calculus HW #6 due Tues. 6 March: Ch. 1.6 # 49, 52, 53, 56

Problem 1.49

(a) Let \( \mathbf{F}_1 = x^2 \mathbf{\hat{z}} \) and \( \mathbf{F}_2 = x \mathbf{\hat{x}} + y \mathbf{\hat{y}} + z \mathbf{\hat{z}} \). Calculate the divergence and curl of \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \). Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.

(b) Show that \( \mathbf{F}_3 = yz \mathbf{\hat{x}} + zx \mathbf{\hat{y}} + xy \mathbf{\hat{z}} \) can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

Problem 1.52

(a) Which of the vectors in Problem 1.15 can be expressed as the gradient of a scalar? Find a scalar function that does the job.

(b) Which can be expressed as the curl of a vector? Find such a vector.

Problem 1.15 Calculate the divergence of the following vector functions:

(a) \( \mathbf{v}_a = x^2 \mathbf{\hat{x}} + 3xz^2 \mathbf{\hat{y}} - 2xz \mathbf{\hat{z}} \).

(b) \( \mathbf{v}_b = xy \mathbf{\hat{x}} + 2yz \mathbf{\hat{y}} + 3zx \mathbf{\hat{z}} \).

(c) \( \mathbf{v}_c = y^2 \mathbf{\hat{x}} + (2xy + x^2) \mathbf{\hat{y}} + 2yz \mathbf{\hat{z}} \).

Problem 1.53 Check the divergence theorem for the function

\[ \mathbf{v} = r^2 \cos \theta \mathbf{\hat{r}} + r^2 \cos \phi \mathbf{\hat{\theta}} - r^2 \cos \theta \sin \phi \mathbf{\hat{\phi}}. \]

using as your volume one octant of the sphere of radius \( R \) (Fig. 1.48). Make sure you include the entire surface. [Answer: \( \pi R^4 / 4 \)]

Problem 1.56 Compute the line integral of

\[ \mathbf{v} = (r \cos^2 \theta) \mathbf{\hat{r}} - (r \cos \theta \sin \theta) \mathbf{\hat{\theta}} + 3r \mathbf{\hat{\phi}} \]

around the path shown in Fig. 1.50 (the points are labeled by their Cartesian coordinates). Do it either in cylindrical or in spherical coordinates. Check your answer, using Stokes' theorem. [Answer: \( 3\pi / 2 \)]

\[ (1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (0, 1, 2) \]