Problem 3.12 Find the potential in the infinite slot of Ex. 3.3 if the boundary at \( x = 0 \) consists of two metal strips: one, from \( y = 0 \) to \( y = a/2 \), is held at a constant potential \( V_0 \), and the other, from \( y = a/2 \) to \( y = a \), is at potential \(-V_0\). See attached worksheet.

Problem 3.13 For the infinite slot (Ex. 3.3) determine the charge density \( \sigma(y) \) on the strip at \( x = 0 \), assuming it is a conductor at constant potential \( V_0 \).

\[
V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5...} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a).
\]  (3.36)

Recall from Ch. 2 that the electric field changes across a charge distribution: \( \mathbf{E} = \frac{\partial \mathbf{E}}{\partial t} = -\nabla V = -\frac{\partial V}{\partial x} \) in this case.

\[
\frac{\partial V}{\partial x} = \frac{4V_0}{\pi} \sum_{n=1,3,5...} \frac{1}{n} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \frac{1}{\pi} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \bigg|_{x=0} = V_0
\]

\[
\frac{\partial V}{\partial x} = -\frac{4V_0}{\pi} \sum_{n=1,3,5...} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \bigg|_{x=0} = 0
\]

\[
\frac{\partial V}{\partial y} = -\frac{4V_0}{\pi} \sum_{n=1,3,5...} \sin\left(\frac{n\pi x}{a}\right) \bigg|_{y=0} = 0
\]

So \( \sigma = -\varepsilon_0 \frac{\partial V}{\partial x} = +\frac{4V_0}{\pi} \sum_{n=1,3,5...} \sin\left(\frac{n\pi x}{a}\right) \bigg|_{y=0} \frac{\varepsilon_0}{a} \).
Separation of Variables: a technique for solving Laplace's equation $\nabla^2 V = 0$

Worksheet for fall E&M Problem 3.12 (p.136)

Find the potential in the infinite slot if the boundary at $x=0$ has two metal strips:

One, from $\left( \frac{a}{2} < y \leq a \right)$, is at potential $-V_0$

The other, from $\left( 0 \leq y \leq \frac{a}{2} \right)$, is held at constant potential $V_0$

First, guess that Laplace's equation $\nabla^2 V = 0$ has solutions of the form

\[ V(x,y) = X(x)Y(y) \]

If so, then the differential equation

\[ \frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0 \]

becomes separable. Substitute (1) into (2) to get

\[ \frac{d^2X}{d^2x^2} + \frac{d^2Y}{d^2y^2} = 0 \]

(3)

Divide (3) by $V = X Y$ and simplify:

\[ \frac{\partial^2 X}{X \partial x^2} + \frac{\partial^2 Y}{Y \partial y^2} = 0 \]

(4)

I argued in class that each term must be constant.

Find solutions to (5) $\frac{\partial^2 X}{\partial x^2} = kX$ and (6) $\frac{\partial^2 Y}{\partial y^2} = -kY$

\[ X = Ce^{-\lambda x} \quad Y = C \sin \lambda y + D \cos \lambda y \]

(3.27)

(5.9)

(5.10)

Substitute (5) and (6) into (1) for the general solution.

\[ V = X Y = e^{-\lambda x} \left( C \sin \lambda y + D \cos \lambda y \right) \]

where $\lambda = \frac{n\pi}{a}$

As we may need both $\sin \frac{n\pi}{a} y$ and $\cos \frac{n\pi}{a} y$ at $V(x,0, y)$

Apply boundary conditions to find undetermined constants.
Now apply BC at \( x = 0 \) to find \( C_n \), as in Ex 3.3 p. 130.

To match \( V(x,0) \):

\[
V(x,0) = \begin{cases} \frac{a}{2} & 0 < x \leq a \\ 0 & x > a \end{cases}
\]

\[
V(x,0) = \frac{a}{2} \quad u = 1, 2, 3, \ldots
\]

Since you can see for these [what odd numbers lead to something like our BC.]

\[
V(x,y) = 2 \sum_{n=1}^{\infty} C_n e^{-\frac{\pi n}{a} x} \sin \frac{\pi n}{a} y
\]

Where, by (3.34), \( C_n = \frac{2}{a} \int_0^a V_0(y) \sin \frac{\pi n}{a} y \, dy \)

\[
C_n = \frac{2}{a} \int_0^a \sin \frac{\pi n}{a} y \, dy = -\frac{2}{\pi n a} \left[ \cos \frac{\pi n}{a} y \right]_0^a
\]

\[
= \frac{2}{\pi n a} \left( \cos \frac{\pi n}{a} - 1 \right) + \frac{2}{\pi n} \cos \frac{\pi n}{2}
\]

\[
C_n = \frac{2 V_0}{\pi n a} \left( \cos \frac{\pi n}{a} - 1 \right) + \frac{2 V_0}{\pi n} \cos \frac{\pi n}{2}
\]

\[
C_n = \frac{2 V_0}{\pi n} \left( 1 + \cos \frac{\pi n}{2} \right)
\]

\[
C_n = \frac{2 V_0}{\pi n} \left( \cos \frac{\pi n}{2} - 1 \right) = 0 \quad n = 3, 7, 11, \ldots
\]

\[
C_n = \frac{2 V_0}{\pi n} \quad \text{for } n = 2, 6, 10, \ldots
\]

\[
V(x,y) = \frac{2 V_0}{\pi n} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{\pi n}{a} x} \sin \frac{\pi n}{a} y
\]
Separation of Variables: Laplace's equation $\nabla^2 V = 0$ in cylindrical coordinates

Worksheet for fall E&M Problem 3.23 (p.145)

Solve Laplace's eqn by separation of variables in cylindrical coordinates, assuming there is no dependence on $z$ (this is cylindrical symmetry).

The Laplacian in cylindrical coordinates is eqn (1.82) p.44:

$$\nabla^2 V = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{2} \frac{\partial^2 V}{\partial z^2}$$

Look for solutions of the form $V(s,\phi) = S(s) \Phi(\phi)$

$$\frac{\partial V}{\partial s} = \frac{\partial S}{\partial s}, \quad \nabla^2 V = \Phi \left( \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial S}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 S}{\partial \phi^2} \right) + \frac{1}{s} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Multiply by $s^2$ and divide by $V = S \Phi$:

$$\frac{\Phi}{s^2} \frac{d}{ds} \left( s \frac{dS}{ds} \right) + \frac{\Phi}{s} \frac{d^2 S}{ds^2} + \frac{d^2 \Phi}{d\phi^2} = 0$$

Both terms must be constant, and they must sum to zero, so the two constants are equal and opposite. Choose $-k^2$ for the $\Phi$ solution so it returns to its original value in one cycle.

Find solutions to (3) and (4)

$$\frac{s d}{ds} \left( s \frac{dS}{ds} \right) = k^2 = C_1$$

1. \[ \frac{d^2 \Phi}{d\phi^2} = C_2 = -k^2 \]

(4) Since $\Phi(\phi + 2\pi) = \Phi(\phi)$, $k$ must be an integer.

$$\frac{d^2 \Phi}{d\phi^2} = -k^2 \Phi \rightarrow \Phi = \Phi_0 e^{k\phi}$$

Show that $S = s^n$ is a solution to (3): what is the relation between $n$ and $k$?

$$\frac{dS}{ds} = s^{n-1}, \quad \frac{d}{ds} \left( s \left[ n s^{n-1} \right] \right) = s^n d\frac{dS}{ds} = n s^n - 1, \quad \frac{d}{ds} \frac{dS}{ds} = k^2$$

Show that for $k$ not 0, $S = A s^k + B s^{-k}$, and for $k=0$ $S = D + C \ln s$.

$$\frac{s}{s} \frac{d}{ds} \left( s \frac{dS}{ds} \right) = 0$$

$$\left( s \frac{dS}{ds} \right) = \text{constant} \rightarrow s \frac{dS}{ds} = S = \int C \frac{ds}{s} = C \ln s + D$$

What is $\Phi$ for the $k=0$ case? Put it all together into a general solution.

Then apply it to problem 3.24.

$$\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \rightarrow \frac{d\Phi}{d\phi} = \text{constant} = a \rightarrow \Phi = b$$

$$\Phi = \int a d\phi = a \phi + b \rightarrow \Phi = b$$

**Solution:** $V = S \Phi = C \ln s + D + \sum_{k=1}^{\infty} \left( A s^k + B s^{-k} \right) e^{ik\phi}$
Problem 3.23 Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on $z$ (cylindrical symmetry). [Make sure you find all solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

see attached worksheet. Solution:

$V = C \ln s + D + \sum_{k=1}^{\infty} \left( A_k s^k + B_k s^{-k} \right) e^{ik \phi}$

$a = 0$

Problem 3.24 Find the potential outside an infinitely long metal pipe, of radius $R$, placed at right angles to an otherwise uniform electric field $E_0$. Find the surface charge induced on the pipe. [Use your result from Prob. 3.23.]

This is much like Ex. 3.8, p. 141

The conducting pipe is an equipotential.

Far from the pipe, $V \rightarrow -E_0 x + C$

(i) $V(s=\infty) = -E_0 s \cos \phi$

(ii) We can set $V(s=R) = 0$

We must fit these boundary conditions to the solution of the Laplacian in cylindrical coordinates, above:

$V = C \ln s + D + \sum_{k=1}^{\infty} \left( A_k s^k + B_k s^{-k} \right) \cos k \phi - \frac{1}{2} s^k \sin k \phi$

$C$ and $D = 0$ because $V = 0$ at $s=R$ - no constant terms.

$k = 0$ because of the orientation of the $E$ field - no sine.

(i) And the only wave number is $k = 1$, since $V \propto \cos \phi$

so $V = \left( a s + \frac{b}{s} \right) \cos \phi$ where $1$ combined $a = \frac{4}{2}, b = Bc$

BC (i) $V(s=R) = 0 = (a R + \frac{b}{R}) \cos \phi \rightarrow a R^2 = -b$

B (i) $V(s=R) = (a s + \frac{b}{s}) \cos \phi \approx a s \cos \phi = -E_0 s \cos \phi$

$a = -E_0 \rightarrow b = +E_0 R^2$
\[ V = (q s + \frac{b}{3}) \cos \phi = (-E_0 s + \frac{E_0 R^2}{s}) \cos \phi \]

\[ V = E_0 s \left( -1 + \frac{R^2}{s^2} \right) \cos \phi \]

\[ \tau = -E_0 \frac{\partial V}{\partial s} \bigg|_{s=R} = -E_0 E_0 \cos \phi \frac{2}{s} \left[ s \left( -1 + \frac{R^2}{s^2} \right) \right] \]

\[ \frac{2}{s} \left[ -s + \frac{R^2}{s} \right] = -1 - \frac{R^2}{s^2} \]

\[ \tau = +E_0 E_0 \cos \phi \left( 1 + \frac{R^2}{s^2} \right) \bigg|_{s=R} = E_0 E_0 \cos \phi \left( 1 + 1 \right) \]

\[ \tau = 2 E_0 E_0 \cos \phi \]