Derive $T'$ for Lorentz transformation between $x$-coordinate in rest frame $S$ and $x'$ coordinate in frame $S'$ moving with speed $v$ in $x$ direction.

\[ T' = \gamma \left[ T (x-vt) + vt' \right] = \gamma^2 (x-vt) + \gamma vt' \]

\[ vt' = \gamma^2 (x-vt) + vt \]

\[ t' = \frac{T}{\gamma} + \frac{x}{\gamma v} = \gamma \left[ t + x \left( \frac{1}{\gamma^2} \frac{1}{v^2} \right) \right] \]

This transformation must work for a flash of light at the origin in either frame.

\[ x^2 + y^2 + z^2 = c^2 t^2 \]

Move frame $x'^2 + y'^2 + z'^2 = c^2 t'^2$. (Note $y' = y$, $z' = z$)

Substitute (1.13) and (1.15) into (1.17):

\[ T (x-vt)^2 + y^2 + z^2 = c^2 T^2 \left[ t + x \left( \frac{1}{\gamma^2} \frac{1}{v^2} \right) \right]^2 \]
(1) \[ y^2(x^2-2vtx+\varepsilon^2t^2)+y^2+z^2 = c^2 \gamma^2 \left(\frac{1}{c^2} \gamma^2 \left(\frac{1-\varepsilon^2}{\gamma^2}\right)^2 \right) + 2\frac{c}{\gamma} \frac{(1-\varepsilon^2)}{\gamma^2}\] 

(2) This must transform to \((1-\varepsilon^2) x^2+y^2+z^2 = c^2 \gamma^2\)

Let's just match the \(x^2\) terms in (1) and (2)

\[ x^2 = x^2 - c^2 \gamma^2 \frac{x^2}{\gamma^2} \left(\frac{1-\varepsilon^2}{\gamma^2}\right)^2 \]

\[ 1 = \gamma^2 \left[ 1 - \frac{c^2}{\gamma^2} \left(\frac{1-\varepsilon^2}{\gamma^2}\right)^2 \right] = \gamma^2 \left[ 1 + \frac{1}{\beta^2} \left(\frac{1-\varepsilon^2}{\gamma^2}\right)^2 \right] \]

or, better

\[ 1-\gamma^2 = \frac{-c^2}{\gamma^2} \left(\frac{1-\varepsilon^2}{\gamma^2}\right)^2 = -\frac{1}{\beta^2} \frac{(1-\varepsilon^2)^2}{\gamma^2} \]

\[ 1 = \frac{-\gamma^2}{\beta^2 \gamma^2} \]

\[ \beta^2 \gamma^2 = \gamma^2 - 1 \]

\[ 1 = \gamma^2 \left(1-\beta^2\right) \rightarrow \gamma^2 = \frac{1}{1-\beta^2} \]

\[ \gamma = \sqrt{\frac{1}{1-\beta^2}} = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}} \]
Consider two inertial reference frames. When an observer in each frame measures the following quantities, which measurements made by the two observers must yield the same results? Explain your reason for each answer.

(a) The distance between two events  
(b) The value of the mass of a proton  
(c) The speed of light  
(d) The time interval between two events  
(e) Newton’s first law  
(f) The order of the elements in the periodic table  
(g) The value of the electron charge

\[ L = \frac{L_p}{\sqrt{1 - \frac{v^2}{c^2}}} \]

H. A. Lorentz suggested 15 years before Einstein’s 1905 paper that the null effect of the Michelson-Morley experiment could be accounted for by a contraction of that arm of the interferometer lying parallel to Earth’s motion through the ether to a length

\[ L = \frac{L_p}{\sqrt{1 - \frac{v^2}{c^2}}} \]

Brownian expansion:

\[ \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = (1 - \frac{1}{2} \frac{v^2}{c^2})^{-\frac{1}{2}} \]

\[ = 1 + \frac{1}{2} (-\frac{v^2}{c^2}) + \frac{1}{2} (-\frac{1}{2}) \frac{1}{2} (-\frac{v^2}{c^2})^2 + \ldots \]

\[ = 1 - \frac{1}{2} \frac{v^2}{c^2} - \frac{1}{8} \frac{v^4}{c^4} + \ldots \]

Since \( v \ll c \)

approximate \( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \approx 1 - \frac{1}{2} \frac{v^2}{c^2} \)

then \( (1 - \frac{1}{2} \frac{v^2}{c^2}) = \frac{1}{2} \frac{v^2}{c^2} \)

\[ L = \frac{L_p}{\sqrt{1 - \frac{v^2}{c^2}}} \]

Length of interferometer arm \( L_p = 11 \text{ m} \)

Speed of Earth "through ether": \( v = 3 \times 10^4 \text{ m/s} \)

\[ \frac{v}{c} = \frac{3 \times 10^4}{3 \times 10^8} = 10^{-4} \]

\[ (1 - \frac{1}{2} \frac{v^2}{c^2}) = \frac{1}{2} \frac{v^2}{c^2} = \frac{1}{2} \times 10^{-8} \]

\[ \Delta L = \frac{L_p}{\sqrt{1 - \frac{v^2}{c^2}}} = 11 \text{ m} \left( \frac{1}{2} \times 10^{-8} \right) = 5.5 \times 10^{-8} \text{ m} \]

"Shrinkage" \( \Delta L = 5.5 \times 10^{-8} \times 550 \text{ atomic diam} = 5.5 \times 10^{-10} \text{ m/atomic diameter} \)
1-49. Frames $S$ and $S'$ are moving relative to each other along the $x$ and $x'$ axes. They set their clocks to $t = t' = 0$ when their origins coincide. In frame $S$, event 1 occurs at $x_1 = 1 \, c \cdot y$ and $t_1 = 1 \, y$ and event 2 occurs at $x_2 = 2.0 \, c \cdot y$ and $t_2 = 0.5 \, y$. These events occur simultaneously in frame $S'$. (a) Find the magnitude and direction of the velocity of $S'$ relative to $S$. (b) At what time do both of these events occur as measured in $S$? (c) Compute the spacetime interval $\Delta s$ between the events. (d) Is the interval spacelike, timelike, or lightlike? (e) What is the proper distance $L_p$ between the events?

1-50. Do Problem 1-49 parts (a) and (b) using a spacetime diagram.

(a) Time dilation $\Delta t' = \Delta t - \frac{\gamma}{c^2} \Delta x$

Simultaneous in $S'$. $\Delta t' = 0 \rightarrow \Delta t' = 0 \rightarrow \gamma \Delta t = \frac{\gamma}{c^2} \Delta x$

Solve for $\gamma = \frac{\Delta t - \Delta t'}{\Delta t'} = \frac{0.5 \, y}{\Delta t'} = \frac{\Delta t'}{c^2}$

(b) At what time do these events occur as measured in $S'$?

$\Delta t' = \frac{(t' - \frac{\gamma}{c^2} \Delta x)}{c^2}$

$\Delta t' = \frac{(1.73 \, y)}{c^2}$

This was for $(x, t) = (1, 0)$. Same for $(x', t') = (2, \frac{1}{2})$

(c) Spacetime interval $\Delta s = \sqrt{(\Delta x)^2 - (\Delta ct)^2} = \sqrt{1^2 - \frac{1}{4}^2} = \sqrt{\frac{15}{16}} = 0.87 \, y$

(d) $\Delta x = 1 \ge (\Delta ct = \frac{\Delta t}{2})$ so the interval is spacelike. Not causally connected events.

(e) $L_p = \Delta s = 0.87 \, c \cdot yr$
The total energy of a particle is twice its rest energy. (a) Find $\gamma \gamma$ for the particle.

Show that its momentum is given by $p = (3)^{1/2} \gamma \gamma$.

\[ E = \gamma m c^2 \rightarrow \gamma \gamma = \frac{E}{m c^2} \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \] and \[ \frac{u}{c} = \sqrt{1 - \frac{u^2}{c^2}} = \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}} \]

\[ \frac{1}{\gamma^2} = 1 - \frac{u^2}{c^2} \Rightarrow \frac{u}{c} = \sqrt{\frac{1}{2}} = 0.87 \]

\[ \frac{E}{m c^2} = \frac{p}{c} \]

\[ (p c)^2 + (m c^2)^2 = E^2 \]

\[ (p c)^2 = E^2 - (m c^2)^2 = (2 m c^2)^2 - (m c^2)^2 \]

\[ = (2 m c^2)^2 - (m c^2)^2 = 3 (m c^2)^2 \]

\[ p c = \sqrt{3} m c^2 \]
A synchronous satellite "parked" in orbit over the equator is used to relay microwave transmissions between stations on the ground. To what frequency must the satellite's receiver be tuned if the frequency of the transmission from Earth is exactly 9.375 GHz? (Ignore all Doppler effects.)

\[ f_0 = \text{current frequency from Earth} \]

**Find the radius and } g \text{ for geosynchronous orbit: } T = 1 \text{ day}**

\[ F = \frac{G M m}{r^2} = m \frac{v^2}{r} \implies v^2 = \frac{GM}{r} \]

\[ \implies \frac{4\pi^2}{T^2} = \frac{GM}{r^3} \]

\[ r^3 = \frac{GM^2}{4\pi^2} = \frac{6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \cdot 6 \times 10^{24} \text{ kg} \cdot (1 \text{ day})^2}{4\pi^2} = \frac{36005}{n} \text{ m} \]

\[ r = 4.23 \times 10^7 \text{ m} \]

Then you can find } g \text{ (r) \text{ at the orbit: } g = \frac{GM}{r^2} \text{ from } \text{GM} = \text{mg} \text{ where } M = \text{mass of Earth, } G = \text{grav. constant} \text{ orbital radius}

\[ g = \frac{GM}{r^2} = \frac{6.67 \times 10^{-11} \text{ kg m}^3 \text{ s}^{-2}}{\left(4.23 \times 10^7 \text{ m}\right)^2} \]

\[ \text{Once you have } g(r), \text{ you can use the general relativistic frequency shift: } \Delta f = f_0 - f = \frac{\Delta f}{c^2} \text{ (2.45)} \]

\[ \text{Where } \Delta f = c \text{ (altitude)} = r - R_{\text{Earth}} = 4.23 \times 10^7 \text{ m} - 6.3 \times 10^6 \text{ m} = 3.59 \times 10^7 \text{ m} \]

\[ \Delta f = 9.375 \text{ GHz} \left(0.221 \frac{\text{ m}}{\text{s}}\right) \left(3.59 \times 10^7 \text{ m} \right) = 8.36 \times 10^{-10} \text{ GHz} \]

\[ f = f_0 - \Delta f : \text{ The frequency is LOWER by a part in a BILLION} \]
In a simple thought experiment, Einstein showed that there is mass associated with electromagnetic radiation. Consider a box of length $L$ and mass $M$ resting on a frictionless surface. At the left wall of the box is a light source that emits radiation of energy $E$, which is absorbed at the right wall of the box. According to classical electromagnetic theory, this radiation carries momentum of magnitude $p = E/c$. (a) Find the recoil velocity of the box such that momentum is conserved when the light is emitted. (Since $p$ is small and $M$ is large, you may use classical mechanics.) (b) When the light is absorbed at the right wall of the box, the box stops, so the total momentum remains zero. If we neglect the very small velocity of the box, the time it takes for the radiation to travel across the box is $\Delta t = \frac{L}{c}$. Find the distance moved by the box in this time. (c) Show that if the center of mass of the system is to remain at the same place, the radiation must carry mass $m = \frac{E}{c^2}$.

(a) $P_{\text{light}} = E = -(P_{\text{box}} - M \cdot v) \Rightarrow v = \frac{E}{M} = \text{recoil speed of box}$

(b) $\Delta x = v \Delta t$, $\Delta t = \frac{L}{c}$

$\Delta x = L \cdot \frac{L}{Mc} = \frac{EL}{Mc^2} = \text{distance box moves before light is absorbed}$

(c) Say radiation of mass $m$ is emitted from the left side of the box at $x = -\frac{L}{2}$. Then the center of mass of the box is at (originally)

$x_{cm} = \frac{\sum \text{m}_i \cdot x_i}{\sum \text{m}_i} = \frac{M \cdot 0 + m \left(\frac{L}{2}\right)}{m + M}$

When the radiation is absorbed on the right side of the box, the box has moved a bit, and the center of mass is now at:

$(M + m) \cdot x_{cm} = M \left(\frac{L}{2} - \Delta x\right) + m \left(\frac{L}{2} - \Delta x\right) = -m \frac{L}{2} = (M + m) \cdot x_{cm}$

$m \frac{L}{2} + m \frac{L}{2} = M \Delta x + M \Delta x = mL$

$M \Delta x = m (L - \Delta x)$

Since this is an internal process, the center of mass cannot move: $x_{cm} = x_{cm}$. Solve for $m$:

$M \Delta x = m (L - \frac{EL}{Mc^2})$

$M \frac{m \cdot EL}{Mc^2} = mL (1 - \frac{E}{Mc^2}) \Rightarrow \frac{E}{c^2} \approx m (1 - \frac{1}{2})$

$\Rightarrow 0 \text{ since } E < Mc^2$