

VISUALIZING RIEMANN SURFACES, TEICHMÜLLER SPACES,  
AND TRANSFORMATION GROUPS ON HYPERBOLIC MANIFOLDS  
USING REAL TIME INTERACTIVE  
COMPUTER ANIMATOR (RTICA) GRAPHICS

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THESIS

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## Abstract

Garsia and Rüedy established the existence of conformal embeddings of all Riemann surfaces into  $\mathbb{R}^3$ . Pinkall constructed embeddings for all closed genus-one Riemann surfaces. In this work we present a new construction of embeddings of tori conformally equivalent to Euclidean rectangles. This construction does not rely on the Hopf fibration of  $S^3$ .

We present graphic tools of independent interest to geometers. The real-time interactive computer animators (RTICA) which incorporate these tools and which produce animated illustrations are described. In particular, we have RTICA that illustrate a Bishop and Frenet frame integrator and that illustrate the fundamental polyhedra in hyperbolic 3-space described by Jørgensen.

The production and presentation of RTICA are described as are the techniques for generating *pic* and *Mathematica* illustrations to be embedded into  $\text{\TeX}$  documents.

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In the same way that one can never truly repay one's debt to one's natural family, I do not expect to be able to repay debts incurred while a student. However, I wish to acknowledge and thank those who allowed me to incur these debts.

## Dedication

To all of my family,  
especially to Robert and Virginia.

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