2. Hyperbolic isometries, the uniformization theorem, and the theory of Riemann surfaces

2.1. Groups central to this work

2.1.1. $PSL(2,\mathbb{C})$, $PSL(2,\mathbb{R})$, and $PSL(2,\mathbb{Z})$.

$PSL(2,\mathbb{C})$: We begin with $GL(2,\mathbb{C})$, the *general linear group* of non-singular 2×2 matrices with complex entries. $SL(2,\mathbb{C})$ is the *unimodular* or *special linear group,* that is, the subgroup of $GL(2,\mathbb{C})$ comprised of elements having determinant equal to one. We obtain $PSL(2,\mathbb{C})$ by taking the quotient of $SL(2,\mathbb{C})$ by its two element center,

$$Z_0 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

$$PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/Z_0$$

Following Jørgensen, we identify the upper half-plane (UHP) as the trace parameter space of $PSL(2,\mathbb{C})$. For convenience, we write $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{C})$ to mean $ad-bc = 1$ and $a+d \in \text{UHP} \cup [0,\infty)$, so a choice is made about which element in each coset in $SL(2,\mathbb{C})/Z_0$ will be the coset's representative.

Our model of $\mathbb{H}^3$, hyperbolic 3-space, will be the upper half-space (UHS). We let $\mathscr{M}\ddot{o}b$ denote[2] the isometries of $\mathbb{H}^3$. We can define the action of $PSL(2,\mathbb{C})$ on $\mathbb{H}^3$ in two different ways so that $PSL(2,\mathbb{C})$ is isomorphic to $\mathscr{M}\ddot{o}b$. In both cases we view the elements of $PSL(2,\mathbb{C})$ as linear fractional transformations:

---

[2] $\mathscr{M}\ddot{o}b$ gets its name from the fact that the orientation preserving isometries of hyperbolic 2- or 3-space are those Möbius transformations that preserve the space and its boundary.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{C})$$

$$M(p) = \frac{ap+b}{cp+d}, \text{ for } p \in \mathbb{C} \cup \{\infty\}$$

$$M(p) = (ap+b)(cp+d)^{-1}, \text{ for } p \in \mathscr{H}, \text{ the quaternions}$$

In the first case, $p \in \mathbb{C} \cup \{\infty\}$; the action of M is restricted to the boundary plane of UHS and then extended via the Poincaré extension[3] to all of $\mathbb{H}^3$. For the extension, we view circles in the extended plane as equators of hemispheres in UHS. The images of hemisphere intersections are the intersection of the hemisphere images. These are determined by the action on the hemisphere equators lying in $\mathbb{C}$. This case emphasizes that the action of a hyperbolic isometry is completely determined by its action on the boundary of the hyperbolic space, either $\mathbb{H}^2$ or $\mathbb{H}^3$.

In the second case, $p \in \mathscr{H}$, the quaternions. That is, $p = x + iy + jz + kt : i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, etc. We identify $\mathscr{H}$ with $\mathbb{R}^4$ and M acts on all of $\mathbb{R}^4$ by $M(p) = (ap+b)(cp+d)^{-1} = (ap+b)(\bar{p}\bar{c} + \bar{d})|cp+d|^{-2}$. It is perhaps surprising that the k-component is suppressed by the action of $PSL(2,\mathbb{C})$ on the quaternions. That is, for $q = x + iy + jz, M(q) = (aq+b)(\bar{q}\bar{c} + \bar{d})|cq+d|^{-2} = \alpha + i\beta + j\gamma + k\delta$, it will always be the case that $\delta = 0$ for every $M \in PSL(2, \mathbb{C})$ and for every $q \in \mathbb{R}^3$. The action of $M(q)$ maintains the k-component's zero value[4]. If we view the quaternions as a trivial $\mathbb{R}^3$ vector bundle over its fourth, or k-component, then the action of $PSL(2,\mathbb{C})$ preserves the fiber over zero. It is within this fiber that our model of $\mathbb{H}^3$ lies. Either way that we define the action of $PSL(2,\mathbb{C})$ on $\mathbb{H}^3$, we obtain the same hyperbolic isometries.

---

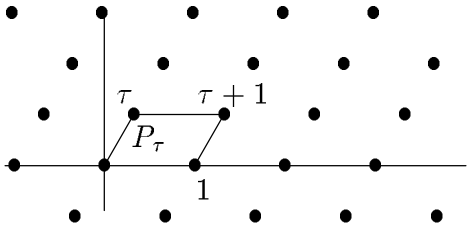
[3] There is a more complete description of the Poincaré extension in section 2.3.2 below.

[4] As Magnus points out in [Ma], the k-component of $M(q)$ is precisely the imaginary part of the determinant of M and we know that this determinant, $ad - bc$, is 1. In particular, it is real.

$PSL(2,\mathbb{R})$: $PSL(2,\mathbb{R})$ preserves the upper half-plane, UHP, our model for $\mathbb{H}^2$. Poincaré showed in [P1] that the action of $PSL(2,\mathbb{R})$ preserves the hyperbolic metric in UHP and that $PSL(2,\mathbb{R})$ is isomorphic to the group of isometries of $\mathbb{H}^2$. One can view $\mathbb{H}^2$ as embedded in $\mathbb{H}^3$ by taking any upper half-plane orthogonal to the boundary plane as a model for $\mathbb{H}^2$.

$PSL(2,\mathbb{Z})$: We now view UHP not as a model of the hyperbolic plane, but rather as the parameter space for closed genus one Riemann surfaces. Given any τ in the upper half-plane, we consider the lattice $\Lambda_\tau = \mathbb{Z}+\tau\mathbb{Z}$. The quotient $\mathbb{C}/\Lambda_\tau$ is a torus. Since the inherited metric is Euclidean, $\mathbb{C}/\Lambda_\tau$ is often referred to as a *flat torus.* The *fundamental domain* is the parallelogram P_τ with vertices $\{0, 1, \tau, \tau+1\}$. The group G, generated by the two transformations, $g_1(z) = z+1$ and $g_2(z) = z+\tau$, is isomorphic to the first fundamental group of the torus, $\mathbb{C}/\Lambda_\tau$. The parallelogram, P_τ *tiles* the plane under the action of G, that is under the action of G, the plane is completely covered by copies of P_τ, with overlapping occurring only at the boundaries of the parallelograms.

(2.1.1.1)



Λ_τ, lattice generated by τ and 1.

The abstract torus, $\mathbb{C}/\Lambda_\tau$, has conformal structure obtained by identifying opposite edges of the parallelogram P_τ. If τ is pure imaginary, then P_τ is a rectangle. A line segment in a parallelogram, P_τ, will be a closed loop on the flat torus, $\mathbb{C}/\Lambda_\tau$, if and only if the segment joins two points on opposite edges of the parallelogram that are identified. A necessary condition for two flat tori to be conformally equivalent, is that line segments in the corresponding (underlying) parallelograms must either close on each of these tori or not close on the tori. For instance, any line meeting the x-axis at an angle of $\frac{\pi}{2}$ will close up on $\mathbb{C}/\Lambda_{1+i}$. Such a line is parallel to the lateral edges of the parallelogram, P_{1+i}, and will intersect the boundary of the parallelogram at identified points. On the other hand, the same line in the plane will wind infinitely many times about the torus $\mathbb{C}/\Lambda_{\sqrt{2}+i}$, densely covering the torus and never closing up. The line will pass through an infinite number of copies of the parallelogram $P_{\sqrt{2}+i}$. However, among the infinite number of intersections with the boundaries of the parallelograms, no two intersection points will be identified. This is one indication that the conformal structures of $\mathbb{C}/\Lambda_{1+i}$ and $\mathbb{C}/\Lambda_{\sqrt{2}+i}$ are not equivalent.

The complex number τ that determines the lattice Λ_τ is called the *modulus* of the flat torus $\mathbb{C}/\Lambda_\tau$. The group $PSL(2,\mathbb{Z})$ is called the *modular group.* The action of $PSL(2,\mathbb{Z})$ on UHP identifies all moduli, τ, for flat tori which have equivalent conformal structures. That is to say, the orbit of $\tau \in$UHP under the action of $PSL(2,\mathbb{Z})$ is an equivalence class of the moduli for all flat tori which have equivalent conformal structures. Let $a, b \in$ UHP and P_a and P_b be the parallelograms with vertices $\{0, 1, a, a+1\}$ and $\{0, 1, b, b+1\}$. Then the flat tori, $\mathbb{C}/\Lambda_a$ and $\mathbb{C}/\Lambda_b$, are conformally equivalent if and only if there is a $g \in PSL(2,\mathbb{Z})$ such that $g(a) = b$. We may view UHP as the parameter space of all flat tori, that is, as the Teichmüller space of closed genus one Riemann surfaces, which we denote as T(1,0).

We note that the groups $PSL(2,\mathbb{C})$ and $PSL(2,\mathbb{R})$, as defined here, are transformation groups acting on $\mathbb{H}^3$ and $\mathbb{H}^2$, respectively. On the other hand, the action of $PSL(2,\mathbb{Z})$ is on the *parameter* space of one type of Riemann surface. We investigate the action that

$PSL(2,\mathbb{Z})$ has on T(1,0) next.

2.1.2. The action of $PSL(2,\mathbb{Z})$ preserves conformal structures

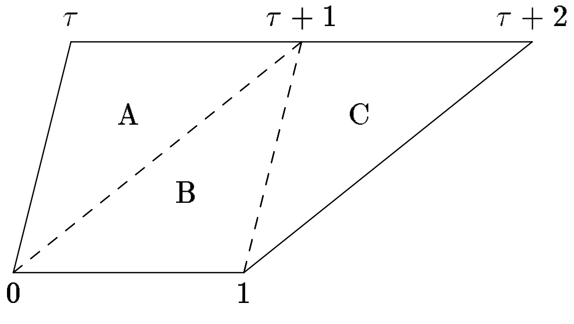
The quotient space T(1,0)/$PSL(2,\mathbb{Z})$ contains a single point for each closed genus one Riemann surface with a given conformal structure. This quotient space is referred to as the *Riemann space* for closed genus one Riemann surfaces, or closed flat tori. It is conformally equivalent to the shaded region in the figure on the following page.

The following two transformations generate[5] $PSL(2,\mathbb{Z})$:

$$g_1(z) = z + 1 \text{ and } g_2(z) = -\frac{1}{z}.$$

How is the action of g_1 on $\tau \in$ UHP translated to the conformal structure of the associated flat torus? Let $\tau \in$ UHP and let us consider the two parallelograms P_τ and $P_{\tau+1}$. P_τ has vertices $\{0, 1, \tau, \tau+1\}$ and $P_{\tau+1}$ has vertices $\{0, 1, \tau+1, \tau+2\}$. P_τ and $P_{\tau+1}$ can be obtained from each other by cutting along a diagonal and gluing along previously identified sides as shown in the figure below.

(2.1.2.1)



$$A \cong C$$

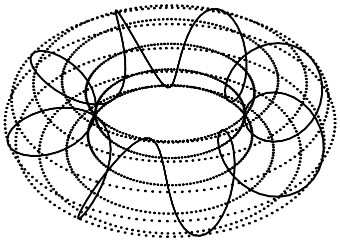
$$P_\tau = A + B \qquad P_{\tau+1} = B + C$$

P_τ conformally equivalent to $P_{\tau+1}$.

---

[5] See, for example, [F, pp79-81].

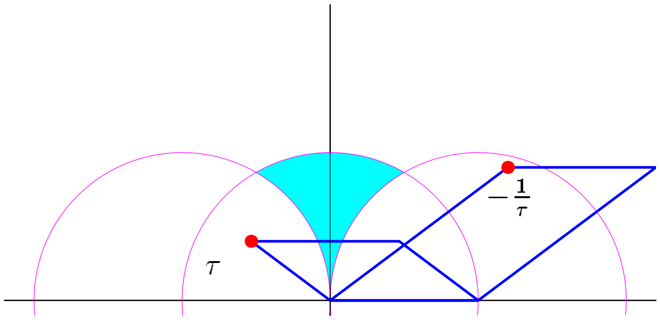
It is known[6] that all closed tori embed conformally in $\mathbb{R}^3$. Consider $\mathscr{T}$, a conformal embedding of P_τ in $\mathbb{R}^3$ as a torus. The new edge in $P_{\tau+1}$ created by the diagonal cut made in P_τ corresponds to an unknot winding once around the torus $\mathscr{T}$. In general, the edge of a parallelogram, $P_{\tau+n}$, will correspond to a coiling unknot winding n times about the torus $\mathscr{T}$ in $\mathbb{R}^3$, shown in the schematic drawing below.



(2.1.2.2)

Coiling unknot corresponds to line segment connecting origin with $\tau + 8$.

Now consider the two parallelograms P_τ and $P_{-\frac{1}{\tau}}$. These two parallelograms are similar. To see this, note that $\arg(-\frac{1}{\tau}) = \pi - \arg(\tau)$ and that the corresponding side ratios of the two parallelograms are $1 : |\tau|$ and $\frac{1}{|\tau|} : 1$.



(2.1.2.3)

P_τ and $P_{-\frac{1}{\tau}}$ are similar parallelograms.

Shaded region is model for Riemann space, isomorphic to T(1,0)$/PSL(2,\mathbb{Z})$.

---

[6] Garsia established in 1961 [G1] that embeddings in $\mathbb{R}^3$ are possible for all conformal types of closed Riemann surfaces.

Since each element of $PSL(2,\mathbb{Z})$ is a finite composition of the two generators, then, for each element $g \in PSL(2,\mathbb{Z})$, the flat tori $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda_{g(\tau)}$ are conformally equivalent. It is in this way that $PSL(2,\mathbb{Z})$ preserves the conformal structure of flat tori.

2.1.3. Existing conformal models of Riemann surfaces

Conformal models of all Riemann surfaces have existed since 1883. Poincaré described the construction of models for all hyperbolic Riemann surfaces in his first article on Fuchsian transformation groups [P1]. He described the necessary and sufficient conditions on sides and angles of a hyperbolic polygon, $\mathscr{P}$, so that $\mathscr{P}$ is a conformal model of a Riemann surface, W. The condition on the sides is that the sides must be congruent in pairs. This allows them to be identified. With these side identifications comes an induced identification of the polygon's vertices. The vertices of $\mathscr{P}$ that are identified to the same point $v \in W$ are referred to as a *cycle.* The condition on the angles is that the vertices in each cycle must sum to 2π. This leaves neighborhoods of v indistinguishable from neighborhoods of all other points in W.

The side identifications can always be accomplished with hyperbolic isometries that map congruent sides onto each other with the correct orientation. These mappings are referred to as *side-pairing transformations* or *deck transformations.* Consider any hyperbolic polygon, $\mathscr{P}$, that models W, a Riemann surface. The side pairing transformations for $\mathscr{P}$ generate a group, G, which is isomorphic to $\pi_1(W)$, the fundamental group of W. A region that tiles the hyperbolic plane under the action of G is called a *fundamental region* for G. The hyperbolic polygon, $\mathscr{P}$ is a fundamental region of G.

2.1.4. Non-hyperbolic Riemann surfaces.

There are only five Riemann surfaces that are not hyperbolic. The Riemann sphere is the only compact, simply-connected Riemann surface. It has genus zero and is referred to as the *elliptic* Riemann surface. The complex plane is the simply connected covering space of itself, the punctured plane, the open torus (no punctures), and the singly punctured torus. These Riemann surfaces all inherit the plane's flat, Euclidean structure. The four Riemann surfaces with the plane as universal covering space are referred to as *parabolic* surfaces. The only ones among these non-hyperbolic surfaces to not embed trivially into $\mathbb{R}^3$ are the tori. The disc, with the hyperbolic metric, is the universal covering space of all other Riemann surfaces. Except for the disc itself, all hyperbolic Riemann surfaces are conformally equivalent to a hyperbolic polygon with edges identified in pairs. Punctures in a hyperbolic Riemann surface correspond to vertices in the polygon that lie on the boundary of the hyperbolic plane. Punctures in a Riemann surface are referred to as *ideal boundary points.*

2.2. The uniformization theorem

In this section we review Ahlfors' proof of the uniformization theorem in [A2]. We begin with a statement of the uniformization theorem.

Uniformization theorem 2.2.1. *Every simply connected Riemann surface is conformally equivalent to a disc (D), to the complex plane ($\mathbb{C}$), or to the Riemann sphere (S^2).*

A simply connected Riemann surface is referred to as a *universal covering space.* A consequence of the theorem is that no matter how abstract the definition of any Riemann surface, W, the surface must be conformally equivalent to a *fundamental region* of one of these three simply connected surfaces. In all but a few exceptional cases, the fundamental region can be constructed as a polygon lying in the universal covering space of W. The theorem guarantees a relationship between relatively abstract objects and very concrete, realizable ones. Of course, reduction to these cases does not always simplify the matters we are addressing here.

2.2.1. Classifying Riemann surfaces.

Let W be a simply connected Riemann surface. If W is open, i.e., not compact, and admits a Green's function, $g(p, p_0)$, for each $p_0 \in W$, then it is conformally equivalent to the open disc, D, and we say W is *hyperbolic.* Constructing the Green's function occupies a major part of the proof, which we discuss below.

If W is open and does not admit a Green's function, then it is conformally equivalent to the plane and we call W *parabolic.* The *elliptic,* simply connected Riemann surface is characterized by being compact and is conformally equivalent to the Riemann sphere.

It turns out that by far the richest and most interesting conformal structures are to be found in the hyperbolic context. As noted above, there are just five of the infinitely many topological types of Riemann surfaces that are not hyperbolic.

2.2.2. Discussion of the proof.

Let W be a simply connected Riemann surface. In each of the three cases, i.e., elliptic, parabolic, and hyperbolic, the theorem's proof is mainly occupied with constructing a function which maps W to an appropriate region in $\mathbb{C} \cup \{\infty\}$. In the hyperbolic case, the Green's function is used to construct the mapping. In the other two cases, a different harmonic function must be constructed to serve the role of the Green's function.

In the hyperbolic case, W is shown to be conformally equivalent to a bounded, simply connected region of $\mathbb{C}$. Then, by the Riemann mapping theorem, we have that W is conformally equivalent to D. Ahlfors constructs the Green's function, $g(p, p_0)$, as the pointwise supremum of a certain family of functions. He shows that $g(p, p_0)$ is harmonic and positive in $W - \{p_0\}$ and that $g(p, p_0) + \log |z(p)|$ has a harmonic extension in a neighborhood of p_0. The first guarantees that $|e^{-(g+ih)}| < 1$ for all h that are harmonic conjugates to g. The second tells us that the growth of g is bound close to $-\log |z(p)|$ near p and allows the following constructions to be extended to neighborhoods of p_0. Choose a covering of $W - \{p_0\}$ by open sets U_α, with each U_α conformally equivalent to an open disc and with p_0 not in any of the U_α. In each U_α, we take a harmonic function, h_α, conjugate to $g(p, p_0)$, to define $f_\alpha = e^{-(g+ih\alpha)}$. The function f_α is analytic on U_α and is determined by g up to a multiplicative constant of modulus 1.

To complete a cover of W, in a neighborhood U_0 of p_0, we can determine a conjugate h_0 of a harmonic extension of $[g(p, p_0) + \log |z(p)|]$. Define $f_0(p) = e^{-(g+ih_0)}$ when $p \neq p_0$ and zero when $p = p_0$. Note that, since $g(p, p_0) > 0$ on W, we have $|f_\alpha| < 1$ on all U_α and $|f_0| < 1$ on U_0.

A topological lemma establishes the ability to globally extend functions that are defined locally in W under the conditions that hold in the present case. By this topological lemma there is a unique function, f, defined on all of W and agreeing with a modified form of f_α on each U_α and with f_0 on U_0. Note that f maps W to a bounded region of $\mathbb{C}$. The result that W is conformally equivalent to the unit disc is established once we show that f is one-to-one.

To establish that f is one-to-one (schlicht), we fix q on W and define

$$F(p) := \frac{f(p,p_0) - f(q,p_0)}{1 - \overline{f(q,p_0)}f(p,p_0)}.$$

Note that $|F| < 1$ on W and $F(q) = 0$. Ahlfors shows that, in fact, $F(p) = e^{i\theta}f(p,q)$ with constant real θ. Further, since $F(p) = 0$ only when $p = q$, we conclude that f is one-to-one.

The theorem is established in this way for the hyperbolic case. After an alternative to the Green's function is constructed, the proof is similar for the parabolic and elliptic cases.

2.2.3. The hyperbolic plane in the parabolic case

As indicated above, the hyperbolic plane is the universal covering space for all but a handful of Riemann surfaces. However, in the two cases of tori, the hyperbolic plane still has a presence, although *not* as the universal covering space. The Teichmüller spaces for flat tori, T(1,0) and T(1,1), are both *isometric* to the hyperbolic plane. The presence of the hyperbolic plane in the one nontrivial case for which it is *not* the universal covering space can cause confusion for those encountering Teichmüller theory for the first time.

Since it is known that Teichmüller spaces, in general, do *not* carry a hyperbolic metric[7], the fact that the Teichmüller spaces T(1,0) and T(1,1) *do* carry such a metric makes them exceptional. Further, they are the only nontrivial Teichmüller spaces that require less than 4 real dimensions to embed. Thus, they are the only ones that lend themselves to easy visualizations. For instance, the Teichmüller space of tori with 2 punctures, T(1,2), is diffeomorphic to a cell in $\mathbb{R}^4$.

---

[7] See, for example, Abikoff [Ab]

2.3. Definitions

2.3.1. Möbius transformation group

Each Möbius transformation acting on $S^2 = \mathbb{R}^2 \cup \{\infty\}$, respectively $S^3 = \mathbb{R}^3 \cup \{\infty\}$, can be realized as a composition of an even number of circle inversions, respectively sphere inversions. For H, any model of hyperbolic 2- or 3-space embedded in S^2 or S^3, the Möbius transformations that preserve the model's boundary and orientation form a group, G, isomorphic to the group of hyperbolic isometries. Any subgroup of G is called a *Möbius transformation group* of H, the model of hyperbolic space. As an example, consider $PSL(2, \mathbb{R})$, which is the largest subgroup of $PSL(2, \mathbb{C})$ that preserves the upper half-plane. $PSL(2, \mathbb{R})$ is isomorphic to the isometry group of the hyperbolic plane for the upper half-plane model of $\mathbb{H}^2$.

2.3.2. Poincaré extension

In [P2] Poincaré described a way to extend the action of $PSL(2,\mathbb{C})$ on $\mathbb{C}\cup\{\infty\}$ to an action on the upper half-space model of $\mathbb{H}^3$. Each circle in $\mathbb{C}$ is viewed as the equator of a hemisphere in the upper half-space.

Let $p \in$UHS and $M \in PSL(2,\mathbb{C})$. The point p can be described (in an infinite number of ways) as the intersection of three hemispheres in UHS. The equators of these hemispheres lie in the xy-plane. The action of M on these circles provides three new circles which determine three hemispheres. The three new hemispheres intersect at a single point, which is defined to be the image of p under the action of M. Poincaré shows that this extended map is well defined and that it preserves the hyperbolic metric in the upper half-space model of $\mathbb{H}^3$.

2.3.3. Isometric circles and hemispheres

Consider a Möbius transformation, $M(z) = \frac{az+b}{cz+d}, ad - bc = 1, a + d \in \text{UHP} \cup [0,\infty)$, which we take as the representative element of one of the two-element equivalence classes in $PSL(2,\mathbb{C})$. For convenience we write $M \in PSL(2,\mathbb{C})$. We view M as acting on the upper half-space model of $\mathbb{H}^3$, as described in the previous paragraph. The locus of all points, z, where $|M'(z)| = 1$, is a Euclidean hemisphere, called the *isometric hemisphere* of M. When the action of M is restricted to $\mathbb{C}$, $|M'(z)|$ is the modulus of the complex number $M'(z)$. In the context of $\mathbb{H}^3$, $|M'(z)|$ is the scalar multiple of the differential map and is called the *conformal factor*. Depending on which context we are in, the modulus, respectively, the conformal factor of the derivative of M is the measure of the local dilation of the transformation. Wherever the derivative has modulus 1, respectively the differential map has conformal factor equal to 1, the transformation is (locally) a Euclidean isometry. At the points inside this circle, respectively this sphere, M is a Euclidean dilation and outside it is a Euclidean contraction.

2.3.4. Properly discontinuous action

Consider G, a group of automorphisms of its domain, U. G is said to be *properly discontinuous* on U if, for every open set $u \in U$, all but finitely many $g \in G$, $u \cap g(u) = \emptyset$. Consider, for example, a polygon $\mathscr{P}$, with edges identified in pairs, which is conformally equivalent to a Riemann surface, W, with universal covering space, U. There are isometries of U that map edges of $\mathscr{P}$ onto the edges with which they are identified. These isometries generate a group G that acts properly discontinuously on U. In fact, in this case $\mathscr{P}$ tiles the universal covering space, U, under the action of the group of isometries, G.

2.3.5. Fundamental regions, Ford and Dirichlet

The *fundamental region* of a Möbius transformation group G, is a region which, together with a portion of its boundary, tiles the plane under the action of G.

When we consider a group of Möbius transformations, G, that acts properly discontinuously on a portion of the complex plane, a fundamental region can be constructed using the isometric circles of the group elements. The intersection of the exteriors of the isometric circles for all the elements of the group is a fundamental region. This is called the group's Ford region, after L. Ford, author of *Automorphic Functions.* [F]

To form a Dirichlet region of G, we fix a point p in $\mathbb{H}^2$. For every $\tau \in \mathbb{H}^2$, we form the orbit of τ, $\mathscr{O}_\tau = \{g(\tau) | g \in G\}$. From each orbit, $\mathscr{O}_\tau$, we take the element whose hyperbolic distance from p is minimal. The set so formed is called the Dirichlet fundamental region. The equivalence of the Ford and Dirichlet regions is discussed in [Ma, pp 70-73].

2.3.6. Ideal boundary points.

Except for six exceptional types, each Riemann surface is conformally equivalent to a hyperbolic polygon which is the fundamental region for G, a group of isometries of the hyperbolic plane. Under the action of G, this polygon tiles the hyperbolic plane. Only isolated boundary points are allowed for a hyperbolic Riemann surface, W. Further, when W is lifted to its universal covering space, the disc D, the boundary points for W coincide with boundary points for D. We refer to these boundary points on W as *ideal boundary points.*

2.3.7. Flat tori.

Let τ be a complex number lying in the upper half-plane and let P_τ be the parallelogram with vertices $\{0, 1, \tau, \tau + 1\}$. The surface obtained by identifying opposite edges of P_τ is topologically a torus. Its inherited metric is Euclidean, so that the surface is referred to as a *flat torus.*

2.3.8. Genus

It is a well known theorem in topology that every orientable surface is a connected sum of n tori. For an orientable Riemann surface, the number n is referred to as the *genus* of the surface.

2.3.9. Signature (p, n) of a Riemann surface.

Let p be the genus of the Riemann surface, W, ignoring its conformal structure, and n the number of punctures. Then (p, n), is called the *signature* of W. The fundamental theorem of surfaces says that two Riemann surfaces are topologically equivalent if and only if they have the same signature.

2.3.10. Markings of a Riemann surface

Consider a Riemann surface, W, which has signature (p, n). W can be cut along $2p + n$ arcs, all based at a point $q \in W$, so that the resulting region with boundary, W', is simply connected. n of these arcs will be open and connect the point q, in turn, with each of the n punctures on W. The remaining $2p$ arcs will be closed loops based at q. Let U_r be a set of arcs based at $r \in W$ and U_s a set of arcs based at $s \in W$, each set cutting W into a simply connected region. We consider U_r and U_s as equivalent if for each closed loop $u \in U_r$, there is one and only one closed loop $v \in U_s$ such that u and v are homotopic to each other in W. A *marking* of W is an equivalence class of sets of arcs on W.

2.3.11. Teichmüller space, T(p, n).

For fixed natural numbers p and n, a point in the Teichmüller space $\mathrm{T}(p, n)$ is the conformal equivalence class of Riemann surfaces of genus p with n points removed. If $n = 0$, the surfaces are referred to as *closed* Riemann surfaces. A point in Teichmüller space, $\mathrm{T}(p, n)$ is a Riemann surface of signature (p, n) together with a marking.

2.3.12. Cyclides of Dupin

A surface of revolution in R^3 generated by a circle, which may or may not have self-intersections, is known as a *cyclide of Dupin.* Each flat torus that is conformally equivalent to a rectangle with opposite edges identified is also conformally equivalent to a cyclide of Dupin.