4. The Bishop and Frenet frame parameters: shaping a space curve by deforming its development

Continuing with the investigations in Chapter 3, we now would wish to conformally embed in $\mathbb{R}^3$ flat tori which are not rectangular. Our strategy is to construct a closed space curve together with its frame and then use this curve and frame to parametrize a tubular surface about the curve. Although we have not yet succeeded in doing this, we describe the RTICA tools developed to approximate the core curves for the tubular surfaces. Using the RTICA, one can reshape the space curve by modifying the frame parameters while viewing the resulting shape in the interactive animation. These tools are of independent interest to geometers.

4.1. Introduction: framings for a space curve γ.

Consider a framable[11] space curve γ. The Bishop and Frenet frames are defined by the following two systems of ordinary differential equations. In this chapter, all differentiation is with respect to, s, the arclength parameter of the space curve γ.

$$\frac{d}{ds}\begin{pmatrix} T \\ \\ M_1 \\ \\ M_2 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ \\ M_1 \\ \\ M_2 \end{pmatrix}$$

$$\frac{d}{ds}\begin{pmatrix} T \\ \\ N \\ \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ \\ N \\ \\ B \end{pmatrix}$$

[11] The Frenet frame requires C^3 curves and the Bishop frame requires that the curves be C^2. The Frenet frame also requires that γ be nondegenerate, that is, γ' and γ'' must be linearly independent. The Bishop frame requires only that γ' be nonzero.

In either case, γ is determined by the relationship $\frac{d\gamma}{ds} = T$. In the Frenet case, N is the principal normal vector and B is the binormal vector. The defining matrices are skew symmetric, and so the frames remain orthogonal. A feature that makes these matrices especially significant is that one of the three off diagonal entries is zero. It might seem that there "ought" to be a third such matrix, since there are three ways to place a zero in the upper or lower triangle. However, two of the three possible matrices actually define equivalent ODE systems. Consider the following system:

$$\frac{d}{ds}\begin{pmatrix} T \\ \\ N_1 \\ \\ N_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix} \begin{pmatrix} T \\ \\ N_1 \\ \\ N_2 \end{pmatrix}$$

As long as we set $a = \kappa$, $b = -\tau$, $N_2 = N$, and $N_1 = B$, then this system is equivalent to the ODE system for the Frenet frame.

Definition 4.1.1. *An orthonormal frame is called* adapted *to the space curve γ if one of the frame's components is the unit tangent vector, T.*

Definition 4.1.2. *A* relatively parallel field *is a normal vector field whose derivative is tangential.*

4.1.1. Frame developments (parameters.)

The Bishop frame parameters, (k_1, k_2), are referred to as the *normal development,* because it is in the normal plane that the first derivative of T and, thus, the curvature of the space curve γ is developed. We will denote the Frenet frame parameters, (κ, τ), as the

Frenet development when we wish to distinguish it from the Bishop frame's normal development. When we speak of both together, we will refer to them simply as *developments.*

Let $\theta = \arg(k_1, k_2) = \operatorname{atan} \frac{k_2}{k_1}$. Let θ' denote the derivative of θ with respect to arc length. Then the relationships between frame parameters are

$$\kappa = \sqrt{k_1^2 + k_2^2}$$

and

$$\tau = \theta' = \frac{k_1 k_2' - k_1' k_2}{k_1^2 + k_2^2}.$$

4.1.2. The Bishop frame.

A surface often associated with the Frenet frame is the cylinder. When the Frenet development is a constant point in the $\kappa\tau$-plane, γ will be a helix lying on a cylinder whose radius is determined by the curvature and angular velocity of the space curve's parametrization. There is an analogous feature for the Bishop frame.

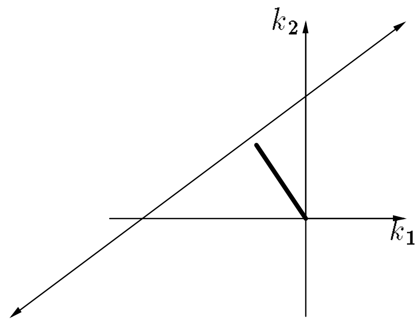
Proposition 4.1.2.1. *When a space curve γ has a normal development that lies on a straight line, then γ will lie on a sphere in $\mathbb{R}^3$, called the osculating sphere.*

Proof: See [Bi2]. □

Proposition 4.1.2.2. *The radius of the osculating sphere, r, is the reciprocal of the distance that the normal development lies from the origin in the frame parameter space.*

Proof: See [Bi2]. □

(4.1.2.1)



$\frac{1}{r}$ is the distance from origin to the normal development

If we express the line on which the normal development lies as $fk_1 + gk_2 = 1$, then the radius of the osculating sphere is $\frac{1}{\sqrt{f^2 + g^2}}$. The center of the osculating sphere is at $P = \gamma + fN_1 + gN_2$. In fact there is, locally, an osculating sphere associated with each point, p, on any normal development. The center and radius of the sphere can be computed from the line tangent to the normal development at p. The computations for all these assertions are carried out in [Bi].

Another feature of the Bishop frame concerns a conjugation rule. Consider a space curve γ. A parametrization of the space curve's normal development gives us a natural parametrization of γ. Let $\delta : (0, 1) \to \mathbb{R}^2$ be the arclength parametrization of the normal development of γ, i.e., $\delta(s) = (k_1(s), k_2(s))$. Let $\mathscr{R}$ denote the mapping from the normal development to the space curve γ.

Let E_2 be a Euclidean rotation in the k_1k_2-plane (the parameter space or normal development space.) Let E_3 be the rotation by the same angle in R^3 about the initial tangent vector of the framing, $\gamma(0) + T(0)$.

Proposition 4.1.2.3. $\mathscr{R}(k_1, k_2) = {\mathrm{E}_3}^{-1} \circ \mathscr{R} \circ \mathrm{E}_2(k_1, k_2)$.

Proof: Recall that $\frac{dT}{ds} = k_1 M_1 + k_2 M_2$. Thus, a rotation in the parameter space (the $k_1 k_2$-plane) corresponds to an identical rotation of the direction of $\frac{dT}{ds}$ in the normal plane, that is, the plane normal to γ at $\gamma(s)$. The consequence of a rotation of the $k_1 k_2$-plane is a simple rotation of the entire system, i.e., curve, frame, and osculating sphere, in R^3. □

Note that as a consequence of this result, when investigating shapes of curves with straight-line normal developments, we can restrict our attention to a single family of parallel lines in the parameter space.

4.2. An RTICA used to construct core space curves

We approximate the map $\mathscr{R}$ in the RTICA using an algorithm[12] for a fourth order Runge-Kutta method. Using the Runge-Kutta algorithm, we can integrate the frame's development to approximate the curve and frame.

$$\mathscr{R} \circ \delta(s) = \{\gamma(s), T(s), M_1(s), M_2(s)\}.$$

This gives us the coordinates of the curve, γ, which we can display on graphics workstations. The frame can be animated to fly along the curve in real-time. When we modify the development, or parameter curve, we can view the results immediately and watch the space curve change shape. A ribbon attached to the space curve reveals the spiraling twist of the frame (related to the parameter τ of the Frenet framing.) As one interacts with the RTICA , one can readily see that specifying a Frenet development that has relatively little twist is a difficult task to accomplish. On the other hand, with the Bishop frame parameters, it is relatively easy to produce a curve that smoothly banks in $\mathbb{R}^3$, that is, makes

[12] We thank Alexei V. Bourd for invaluable assistance given as we developed code for the Runge-Kutta algorithm used in the RTICA described in this Chapter.

turns with relatively little torsion and secondary coiling. Interacting with the RTICA, one can view the computed curve and frame as one attempts to "steer" the space curve into a desired shape.

4.3. Constructing *preconformal* tori

Let U be any relatively parallel field for a framable space curve γ. Bishop shows that $U = \gamma + r\left(\cos\theta M_1 + \sin\theta M_2\right)$, with r and θ fixed. [Bi]

A necessary condition for a torus[13] in $\mathbb{R}^3$ to be conformally equivalent to a flat torus is the following.

Definition 4.3.1. *A map from a parallelogram onto a torus is called* preconformal *if there exist two families of parameter curves on the torus that intersect everywhere with the same angle.*

Let γ be a space curve with an adapted framing, $\{T, M_1, M_2\}$. Let s be the arclength parameter of γ and $0 < \psi \leq 2\pi$

Define

$$\mathscr{T}(s,\psi) = \gamma(s) + r(s)\left(\cos\psi M_1(s) + \sin\psi M_2(s)\right). \tag{4.3.1}$$

If r is constant, then the tube defined above is the envelope of radius r circles in the M_1M_2-plane centered on γ. Points on this circle in the M_1M_2-plane, viewed as attached to the moving frame, trace out the relatively parallel fields. A trace on $\mathscr{T}$ with constant angle ψ is called a *latitude*. The normal plane at a point on γ intersects the surface $\mathscr{T}$ in a circle that we refer to as a *meridian* of $\mathscr{T}$.

[13] When we use the term *torus*, we mean a topological genus one surface in $\mathbb{R}^3$, not necessarily a surface of revolution.

If γ is a circle and M_1 always lies in the plane of that circle, then the definition of the map $\mathscr{T}$ given above is a familiar map of a rectangle onto a torus of revolution. The latitudes and meridians all intersect each other at right angles, so we have an example of a *preconformal* structure on the torus.

Suppose that $\mathscr{T}$ is a torus embedded in $\mathbb{R}^3$ with the space curve γ as its core curve. Suppose further that γ has variable curvature and that $\mathscr{T}$ is parametrized by relatively parallel fields of γ.

Conjecture: 4.3.2. *For a core curve with variable curvature, the structure of $\mathscr{T}$ can be conformally equivalent to a flat torus only if the radius r is not a constant.*

Proposition. 4.3.3. *The conjecture is true for the elliptic case.*

Poof: Consider the ellipse, $0 \le t < 2\pi$.

$$\gamma(t) = \left(\frac{1}{2}\cos t, \sin t\right).$$

Let M_1 be the unit normal lying in the xy-plane, pointing in the inward direction, and M_2 the usual binormal vector. Define

$$\mathscr{T}(t, \phi) = \gamma(t) + R(M_1 \cos\phi + M_2 \sin\phi)$$

We recall considerations discussed in Chapter 3. A prerequisite for $\mathscr{T}$ to be conformally equivalent to a flat torus is that infinitesimal rectangles in the parallelogram P map to infinitesimal congruent figures on $\mathscr{T}$. Recall that for this to be possible for the map $\mathscr{F}$, there must be a vertical contraction on regions of P that map to regions on $\mathscr{T}$ with negative curvature. This was illustrated in figure 3.3.3.2.

The details of the construction in Chapter 3 indicate that, for a given radius r, the greater the curvature of the core curve for $\mathscr{T}$, the greater this contraction must be. If the radius of the torus, $\mathscr{T}$, were constant, then the parameter lines (meridians and latitudes) would not all meet at right angles. Thus, the map could not be conformal. □

Consider the tube about a core curve that is an ellipse, γ, with minor radius, $r = \frac{1}{\kappa}$. The inner equator of the tube coincides with the evolute of γ. We conjecture that this tube can be reparametrized by Möbius transformations in a manner analogous to the process described in Chapter 3, so that the conformal structure of the tube is equivalent to that of a rectangle.

Conjecture. 4.3.4. *A variable radius, $r = \frac{1}{\kappa}$, along with a variable Möbius transformation, will allow us to construct a conformal map from the rectangle, P, to a torus with an elliptic core curve.*

If this reparametrization can be accomplished, one might hope to generalize the process to non-planar core curves.

4.4. Stereographic projection and loxodromes

Stereographic projection of the complex plane $\mathbb{C}$ onto the Riemann sphere with projection point $(0, 0, 1)$, is written

$$z = x + iy \to \begin{pmatrix} \frac{2x}{|z|^2+1} \\ \frac{2y}{|z|^2+1} \\ \frac{|z|^2-1}{|z|^2+1} \end{pmatrix}$$

An alternative way of writing the same projection[14] is as follows

Let $\mathscr{L} = \log |z|$.

$$z = x + iy \to \left(\frac{z}{|z|} \operatorname{sech} \mathscr{L}, \tanh \mathscr{L} \right) \in \mathbb{C} \times \mathbb{R}.$$

[14] This form for stereographic projection was suggested by Richard Bishop.

This form yields a particularly useful way to describe the locus of a loxodrome on the unit sphere. A loxodrome is the image under stereographic projections of a logarithmic spiral lying in $\mathbb{C}$. The defining feature is that the loxodrome on the sphere intersects all parallels at equal angles. Consider such a logarithmic spiral, $z = e^{(1+ib)t}$. The parameter b is a fixed real number and the parameter t ranges from negative to positive infinity. With this notation, a formula for the associated loxodrome is

$$p(t) = (u, v) = (e^{ibt} \operatorname{sech} t, \tanh t).$$

4.5. The Bishop frame for loxodromes

Each development curve that lies on a straight line in the k_1k_2-plane, produces a space curve on a sphere that spirals about a point in a manner similar to a loxodrome. At first glance, it appears that such a space curve might, in fact, be a loxodrome. With the notation introduced above, it is easy to show that this is not the case, that is, we show

Proposition: 4.5.1. *A normal development that lies on a straight line segment and which is parametrized by arc length cannot be the normal development of a loxodrome.*

Of course, since the loxodrome lies on a sphere of constant radius, the loxodrome's normal development must lie on a straight line. The issue is the parameterization of the normal development. Further, this parametrization is not to be confused with the parametrization of the space curve γ.

Proof:

It is enough to show that, for the loxodrome, the derivative of the normal development parameters, (k_1, k_2), with respect to the development's arclength, are not constants. First we compute the Bishop frame for the loxodrome p on the unit sphere. The speed of the curve p is $\dfrac{ds}{dt} = \left|\dfrac{dp}{dt}\right|$.

$$\frac{dp}{dt} = \{(-\operatorname{sech} t \tanh t + ib \operatorname{sech} t)e^{ibt}, \operatorname{sech}^2 t\}$$

$$= \{-\operatorname{sech} t e^{ibt} \tanh t + ib \operatorname{sech} t e^{ibt}, \operatorname{sech}^2 t\}$$

$$= \{-uv + ibu, |u|^2\}$$

$$= \{(-v + ib)u, |u|^2\}$$

Thus

$$\left|\frac{dp}{dt}\right| = \sqrt{|(-v+ib)u|^2 + ||u|^2|^2}$$

$$= \sqrt{(v^2+b^2)|u|^2 + |u|^4}$$

$$= |u|\sqrt{v^2 + b^2 + |u|^2}$$

Further, since $v^2 + |u|^2 = \tanh^2 t + \operatorname{sech}^2 t = 1$,

$$\frac{ds}{dt} = |u|\sqrt{1+b^2} \qquad (4.5.1)$$

Also,

$$T = \frac{dp}{dt}\frac{dt}{ds} = \frac{1}{|u|\sqrt{1+b^2}}\{(-v+ib)u, |u|^2\}$$

$$= \frac{1}{\sqrt{1+b^2}}\left\{(-v+ib)\frac{u}{|u|}, |u|\right\}$$

and, since $\dfrac{u}{|u|} = e^{ibt}$,

$$= \frac{1}{\sqrt{1+b^2}}\{(-v+ib)e^{ibt}, |u|\}$$

Generally, for a curve in a sphere of radius R, we can choose the inward sphere normal N_1 as part of the Bishop frame. Since a loxodrome lies on a sphere, its normal development lies on a straight line. Consider the normal development $(k_1, k_2) = (1, k_2)$, that is, the vertical line $k_1 = 1$. This vertical line produces a space curve that lies on S, the sphere of radius 1. The center of S depends on initial conditions which are unspecified in this discussion. We compute what k_2 must be for $(1, k_2)$ to be the normal development of a loxodrome that lies on S.

So $N_1 = -p \in S^2$. Thus

$$\frac{dT}{ds} = k_1 N + k_2 B = N + k_2 B = -p + k_2 B \tag{4.5.2}$$

Also, from 4.5.1,

$$\frac{dT}{ds} = \frac{dt}{ds}\frac{dT}{dt} = \frac{1}{\sqrt{1+b^2}|u|}\frac{dT}{dt}$$

We now use

$$\frac{d|u|}{dt} = \frac{\mathrm{dsech}\, t}{dt} = -\tanh t \operatorname{sech} t = -v|u|$$

and

$$\frac{dv}{dt} = |u|^2$$

to get $\dfrac{dT}{dt}$.

$$\begin{aligned}\frac{dT}{dt} &= \frac{d}{dt}\left(\frac{1}{\sqrt{1+b^2}|u|}\{(-v+ib)u, |u|^2\}\right)\\ &= \frac{d}{dt}\left(\frac{1}{\sqrt{1+b^2}}(-v+ib)e^{ibt}, |u|\right)\end{aligned}$$

$$= \frac{1}{\sqrt{1+b^2}} \frac{d}{dt} \left\{ (-v+ib)e^{ibt}, |u| \right\}$$

$$= \frac{1}{\sqrt{1+b^2}} \left\{ -\frac{dv}{dt} e^{ibt} + (-v+ib)ibe^{ibt}, \frac{d|u|}{dt} \right\}$$

$$= \frac{1}{\sqrt{1+b^2}} \left\{ -|u|^2 e^{ibt} - ibve^{ibt} - b^2 e^{ibt}, -v|u| \right\}$$

$$= \frac{1}{\sqrt{1+b^2}} \left\{ (-|u|^2 - ibv - b^2)e^{ibt}, -v|u| \right\}$$

$$= \frac{1}{\sqrt{1+b^2}} \left\{ (-|u|^2 - ibv - b^2)\frac{u}{|u|}, -v|u| \right\}$$

$$= \frac{1}{\sqrt{1+b^2}} \left\{ -|u|u - ibv\frac{u}{|u|} - b^2 \frac{u}{|u|}, -v|u| \right\}$$

$$= \frac{|u|}{\sqrt{1+b^2}} \left\{ -u - ibv\frac{u}{|u|^2} - b^2 \frac{u}{|u|^2}, -v \right\}$$

and, so

$$\frac{dT}{ds} = \frac{dt}{ds}\frac{dT}{dt}$$

$$= \frac{1}{\sqrt{1+b^2}|u|} \frac{|u|}{\sqrt{1+b^2}} \left\{ -u - ib^2\frac{v}{b}\frac{u}{|u|^2} - b^2 \frac{u}{|u|^2}, -v \right\}$$

$$= \frac{1}{1+b^2} \left\{ -u - b^2(i\frac{v}{b}+1)\frac{u}{|u|^2}, -v \right\}$$

Now from 4.5.1 we have

$$-p + k_2 B = \frac{dT}{ds}$$

Thus,

$$k_2 B = \frac{dT}{ds} + p = \frac{dT}{ds} + \{u, v\}$$

$$= \frac{1}{1+b^2} \left\{ -u - b^2(i\frac{v}{b}+1)\frac{u}{|u|^2}, -v \right\} + \{u, v\}$$

$$= \frac{1}{1+b^2} \left\{ -u - b^2(i\frac{v}{b}+1)\frac{u}{|u|^2}, -v \right\} + \frac{1}{1+b^2} \left\{ (1+b^2)u, (1+b^2v) \right\}$$

$$= \frac{1}{1+b^2} \left\{ b^2u - b^2(i\frac{v}{b}+1)\frac{u}{|u|^2}, b^2v \right\}$$

$$= \frac{b^2}{1+b^2} \left\{ u - (i\frac{v}{b}+1)\frac{u}{|u|^2}, v \right\}$$

$$= \frac{b^2}{1+b^2}\left\{u(1 - \frac{1}{|u|^2} - i\frac{v}{b|u|^2}), v\right\}$$

and, so

$$|k_2|^2 = \left(\frac{b^2}{1+b^2}\right)^2 \left(|u|^2((1 - \frac{1}{|u|^2})^2 + (\frac{v}{b|u|^2})^2) + v^2\right)$$
$$= (\frac{b^2}{1+b^2})^2 \left(|u|^2(1 - \frac{2}{|u|^2} + \frac{1}{|u|^4} + \frac{v^2}{b^2|u|^4}) + v^2\right)$$

since $|u|^2 + v^2 = 1$,

$$= \left(\frac{b^2}{1+b^2}\right)^2 \left(1 - 2 + \frac{1}{|u|^2} + \frac{v^2}{b^2|u|^2}\right)$$
$$= \left(\frac{b^2}{1+b^2}\right)^2 \left(-1 + \frac{1}{|u|^2}(1 + \frac{v^2}{b^2})\right)$$

We conclude that $\frac{dk_2}{ds}$ is not a constant. Thus, the normal development for the loxodrome is *not* a straight line with arclength parametrization in the k_1k_2-plane. □

In the following Chapter we discuss how the loxodrome is the geometric figure central to the distinction between isometry groups of $\mathbb{H}^2$ and $\mathbb{H}^3$.