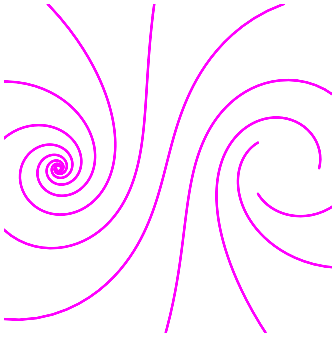
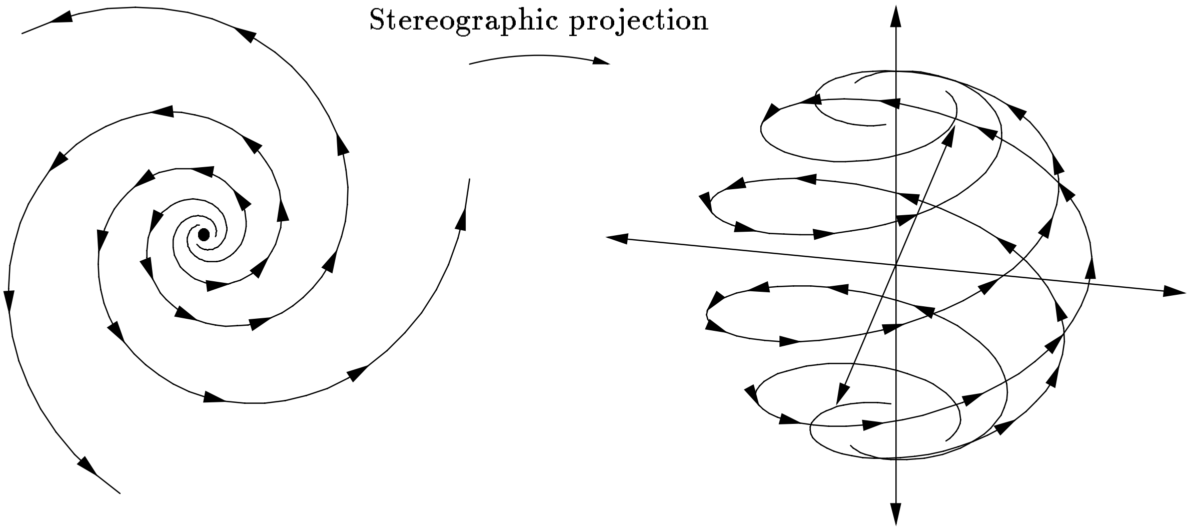
5. Möbius transformations in $\mathbb{H}^2$ and Jørgensen's classification of loxodromic transformations in $\mathbb{H}^3$



The inset picture is a fanciful design made of loxodromes mapped under a Möbius transformation. Loxodromes are important in ship navigation on the earth's surface. A *loxodrome*, also called a *rhumb line*, is the path followed by a ship or airplane that is traveling with a constant compass bearing, that is, along a path that meets all parallels on the globe at a constant angle. While not the shortest distance between two points on a sphere, it is a path that can be followed by a ship's navigator with compass alone. Before the invention of accurate clocks, which are needed to determine the longitude, it was not possible to navigate along geodesics, which are great circles.

The following figure depicts a globe with loxodromes on its surface, the stereographic image of logarithmic spirals in $\mathbb{C}$. The images of flow lines are shown for the loxodromic Möbius transformation, $M(z) = 1.35e^{0.2i}z$.

(5.1)



Logrithmic spirals mapped to loxodromes on a sphere

5.1. Characterizations of Möbius transformations in $PSL(2,\mathbb{C})$

We usually introduce Möbius transformations as the totality of one-to-one, conformal self-mappings of the extended complex plane. This group, $\mathscr{M}\ddot{o}b$, is isomorphic to $PSL(2,\mathbb{C})$. The action of $PSL(2,\mathbb{C})$ on $\mathbb{C}\cup\{\infty\}$ is usually defined as follows. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc = 1$, $a + d \in$UHP$\cup[0,\infty)$ and define the action of M on $z \in \mathbb{C}\cup\{\infty\}$ to be $M(z) = \dfrac{az+b}{cz+d}$. In this case, M is the canonical representative of a coset element in $PSL(2,\mathbb{C})$. For convenience, we use the notation $M \in PSL(2,\mathbb{C})$. Most of the computations that justify statements made in the next three subsections are to be found in Ford [F, pp15-31].

There are four general types of Möbius transformations: elliptic, parabolic, hyperbolic, and loxodromic. We characterize these types in several ways: by the trace of the transformations' matrices, by their actions on certain *Steiner nets,* by configurations of fixed points and isometric circles, and, finally, by Euclidean transformations. Each type is equivalent to one of four elementary Euclidean transformations: rotation, translation, dilatation[15], or spiral dilatation.

5.1.1. The trace

The *trace* of an element in $PSL(2,\mathbb{C})$, $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be: $\operatorname{Tr} M = a + d$. The principal feature of the trace is that it is invariant under conjugation, $N^{-1}MN$, within $PSL(2,\mathbb{C})$.

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[15] A Euclidean dilation, centered at the origin of $\mathbb{C}$, can be expressed as Kz, with $K > 1$. In this section, when we use the term dilatation, we will mean to include the possibility that $K < 1$, that is, that a dilatation could be either a dilation or a contraction.

The possible trace values for a Möbius transformation, M, are:

elliptic	$-2 < \mathrm{Tr}\, M < 2$
parabolic	$\mathrm{Tr}\, M = \pm 2$
hyperbolic	$\mathrm{Tr}\, M < -2$ or $2 < \mathrm{Tr}\, M$
loxodromic	$\mathrm{Im}\mathrm{Tr}\, M \neq 0$

We will discuss the space of trace values and a partition of it due to Jørgensen in the second section of this Chapter.

5.1.2. Fixed points and isometric circles

Fixed points. Consider $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{C})$. Fixed points for $M(z) = \frac{az+b}{cz+d}$ are the roots of the quadratic $cz^2 + (d-a)z - b$. If the discriminant[16] is zero, that is, if $(a+d)^2 - 4 = 0$, then $\mathrm{Tr}\, M = a + d = \pm 2$ and M is parabolic. Having a double fixed point is characteristic of a parabolic Möbius transformation.

Isometric circles. Let M be as above, only now with $c \neq 0$, which is equivalent to ∞ *not* being a fixed point. The locus of all points in $\mathbb{C}$ where M is a local Euclidean isometry is a Euclidean circle. $|M'(z)| = 1$ on the circle $|z - \frac{-d}{c}| = \frac{1}{|c|}$. We call this circle the *isometric circle* of M and denote it as I_M. The isometric circle of M^{-1} is $|z - \frac{a}{c}| = \frac{1}{|c|}$, which we denote by $I_{M^{-1}}$.

A relationship that holds is $M(I_M) = I_{M^{-1}}$. For convenience, we shall refer to both I_M and $I_{M^{-1}}$ as the isometric circles of M. Isometric circles for elliptic transformations intersect at the transformation's fixed points. For parabolic transformations, the isometric circles are tangent at the single fixed point. The isometric circles for hyperbolic transfor-

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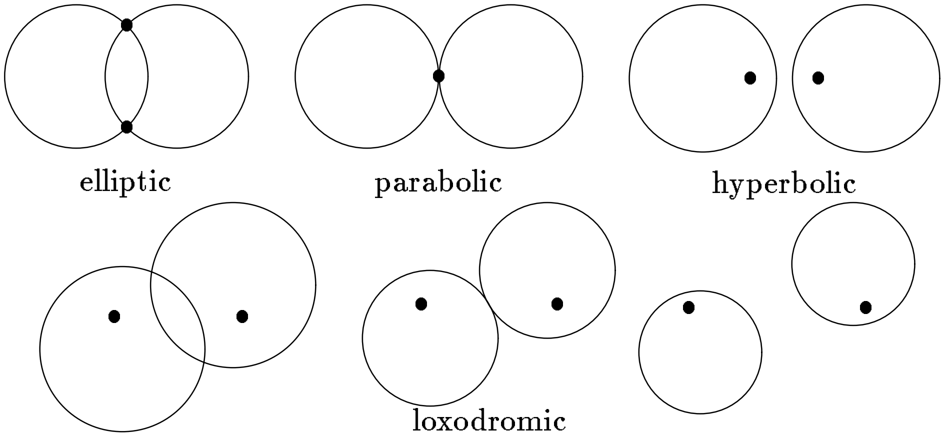
[16] The discriminant for the quadratic $cz^2 + (d-a)z - b$ is $(a-d)^2 + 4bc = (a+d)^2 - 4$, since $ad - bc = 1$. The last expression is equivalent to $(\mathrm{Tr}\, M)^2 - 4$.

mations are disjoint. For loxodromic Möbius transformations, all three configurations of isometric circles are possible.

(5.1.2.1)



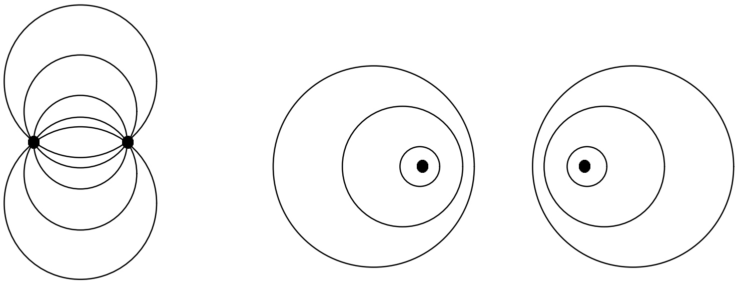
Isometric circle and fixed point configurations.

5.1.3. Steiner nets.

A Steiner net for a surface consists of two orthogonal families of curves that cover the surface, e.g., the latitudes and longitudes on a sphere. The Cartesian coordinate lines on the plane, i.e., the two families of lines parallel, respectively, to the x- and y-axes is another Steiner net. The Steiner nets associated with a Möbius transformation, $M(z)$, contains a family of curves, each of which is preserved by $M(z)$. The second family of curves has its elements permuted by the action of $M(z)$. Elliptic and hyperbolic Möbius transformations have identical associated Steiner nets, which we now construct. Let $\{p_1, p_2\}$ be the fixed points of $M(z)$, which may be either hyperbolic or elliptic. Denote the stereographic projection from the sphere to the extended complex plane by $\mathscr{S}$. Let Q be the family of longitude great circles on the sphere and P the family of latitudes. Define $g(z) = \dfrac{z - p_2}{z - p_1}$. The map g takes the fixed points, $\{p_1, p_2\}$, to the points $\{\infty, 0\}$.

Define $L = g^{-1} \circ \mathscr{S}(P)$ and $H = g^{-1} \circ \mathscr{S}(Q)$. L and H together will be a Steiner net for the extended complex plane because the maps $\mathscr{S}$ and g are conformal and one-to-one. H contains all circles passing through both fixed points p_1 and p_2. L is the family of circles orthogonal to all members of H. Further, the fixed points are inverses[17] in every element of the family L.

(5.1.3.1)



$H = g^{-1} \circ \mathscr{S}(\text{longitudes})$ $L = g^{-1} \circ \mathscr{S}(\text{latitudes})$

Hyperbolic and elliptic families of circles.

Each element of H will be mapped onto itself by a hyperbolic transformation, $M(z)$, with fixed points $\{p_1, p_2\}$. We refer to H as the *hyperbolic family of circles* for $M(z)$ and say that H is *preserved* by $M(z)$. Further, a hyperbolic $M(z)$ will permute the elements of L. On the other hand, if $M(z)$ is elliptic, it will preserve L and permute H. We refer to L as the *elliptic family of circles* for $M(z)$.

The following figures show the Steiner net for elliptic and hyperbolic transformations. Although the nets are identical, arrows indicate the difference in the action that the transformations have on $\mathbb{C}$.

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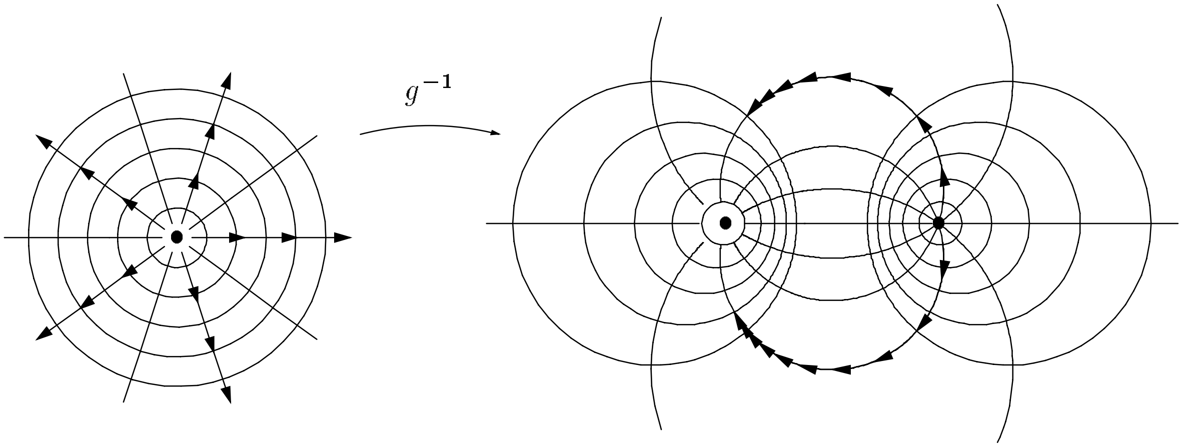
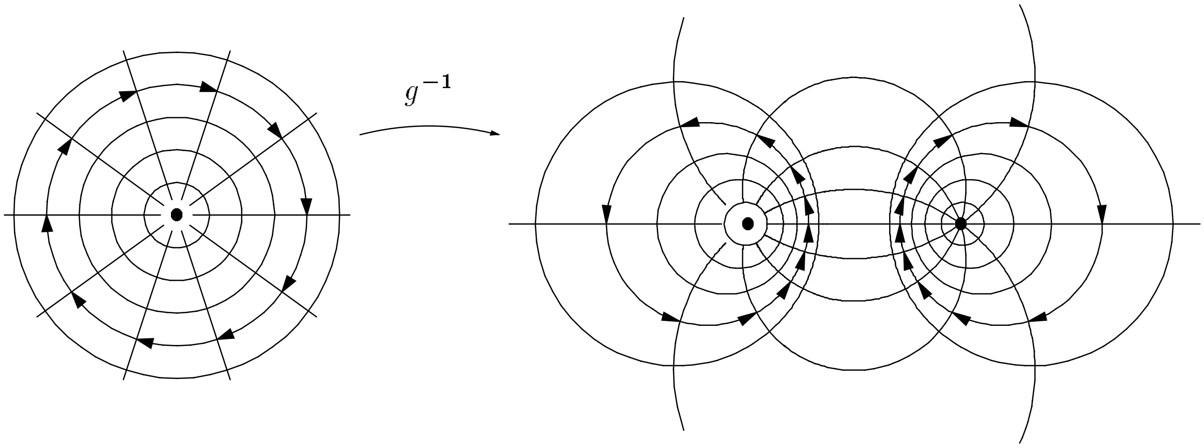
[17] If an element of L is the circle of radius R centered at q, then $R^2 = |(q - p_1)(q - p_2)|$ and q, p_1, and p_2 will be collinear.

(5.1.3.2)



Hyperbolic transformation is conjugate equivalent to a dilatation.

(5.1.3.3)



Elliptic transformation is conjugate equivalent to a rotation.

Let $M(z) = \dfrac{az+b}{cz+d}$ be parabolic with fixed point, $p \neq \infty$, which is equivalent to $c \neq 0$. Let $g(z) = \dfrac{1}{z-p}$. Elementary computations[18] show that

$$\frac{1}{M(z)-p} = \frac{1}{z-p} + c.$$

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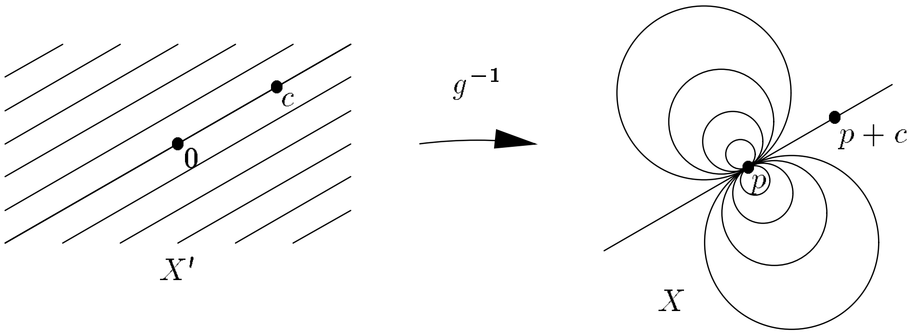
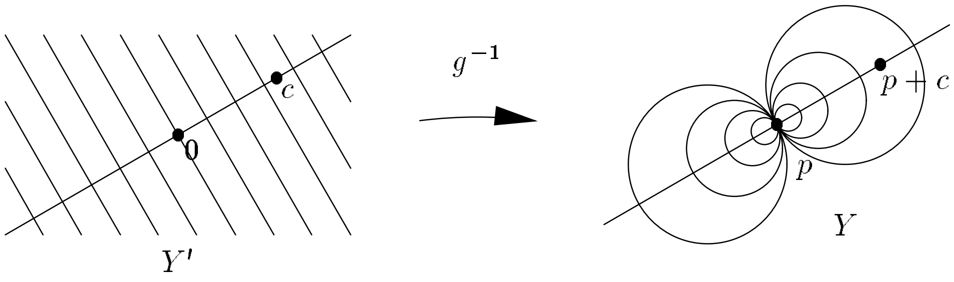
[18] If $c \neq 0$, the fixed point $p = \dfrac{a-d}{2c}$. Computations based on this fact establish the next equation and may be found in Ford,[F, p22].

Let X' be the family of lines parallel to the ray through c from the origin and let Y' be the family of lines orthogonal to those in X'. Then $X = g^{-1}(X')$ and $Y = g^{-1}(Y')$ together form a Steiner net for $M(z)$ with the families X preserved and Y permuted by the action of $M(z)$.

(5.1.3.4)



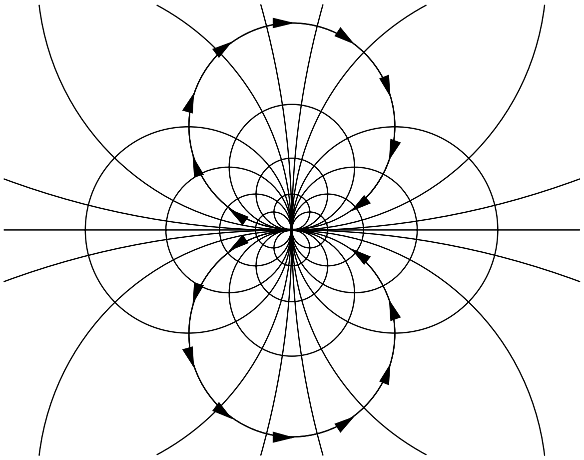
(5.1.3.5)



Two orthogonal families of parallel lines transformed by g^{-1}.

Following is a figure of the Steiner net for a parabolic transformation with arrows indicating the action of the transformation.

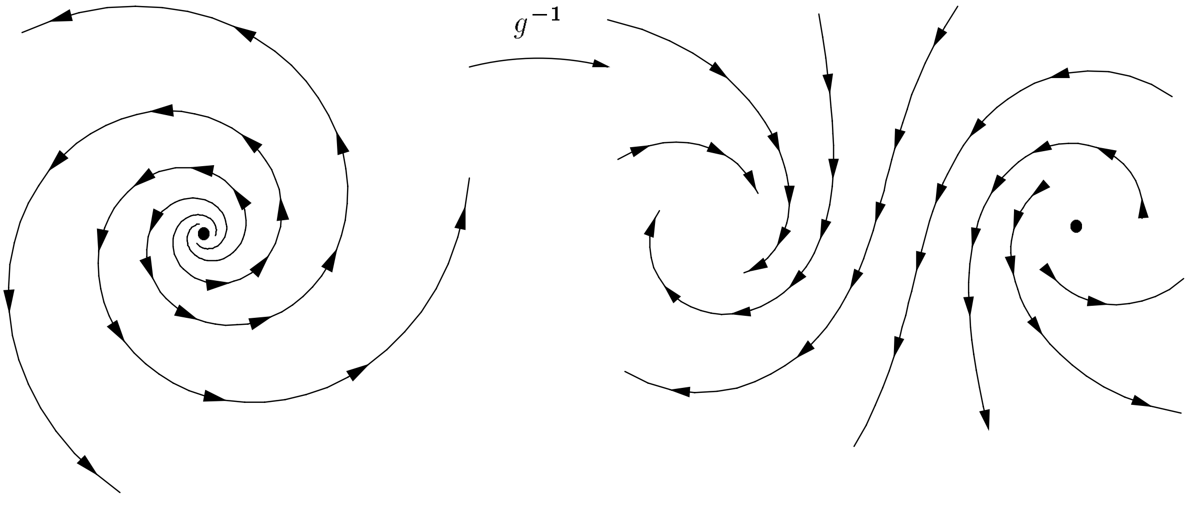
(5.1.3.6)



Steiner net of a parabolic transformation with arrows on flow lines.

Loxodromic case. Now we consider the case when $M(z)$ is a loxodromic Möbius transformation. The family of curves preserved is no longer a family of circles. It is a family of generalized loxodromes, like those shown below.

(5.1.3.7)



Loxodromic transformation is conjugate equivalent to a spiral dilatation.

The curves orthogonal to this family of loxodromes are themselves loxodromes, winding in the opposite direction.

Let $M(z)$ be any type of Möbius transformation other than elliptic. Then one of the fixed points for $M(z)$ is the attractor for all points in $\mathbb{C}$ (save the fixed points) under iteration by $M(z)$. The second fixed point is a repellor under iteration by $M(z)$. If $M(z)$ is parabolic, then the single fixed point attracts in one direction and repels in the other. This can be seen in the parabolic Steiner net illustration above.

5.1.4. Elementary Euclidean motions

Each Möbius transformation is conjugate equivalent to an elementary Euclidean transformation. Let $M \in PSL(2,\mathbb{C})$. Then, there is an element $g \in PSL(2,\mathbb{C})$ such that $g \circ M \circ g^{-1}(z) = E(z)$, where $E(z)$ is one of the following Euclidean transformations: rotation, translation, dilatation, or spiral dilatation. The traces of $E(z)$ and $M(z)$ will be the same in every case.

Consider the case where $M(z)$ has two finite fixed points, p_1 and p_2. Then $E(z) = Ke^{i\theta}z$; $K \in (0,+\infty)$, $0 \leq \theta < 2\pi$ and the required $g(z) = \dfrac{z-p_2}{z-p_1}$. If $M(z)$ is elliptic, then $K = 1$ and $E(z)$ is a rotation of angle θ about the origin. If $M(z)$ is hyperbolic, then $\theta = 0$ and $E(z)$ is a dilatation, actually a dilation when $K > 1$ and a contraction when $K < 1$. If $K \neq 1$ and $0 < \theta < 2\pi$, then $M(z)$ is loxodromic and $E(z)$ is a spiral dilatation. When $M(z)$ has a single fixed point, p, then $E(z) = z + \alpha, \alpha \in \mathbb{C}$ and $g(z) = \dfrac{1}{z-p}$. In this last case, $M(z)$ is parabolic and $E(z)$ is a translation in the direction of α.

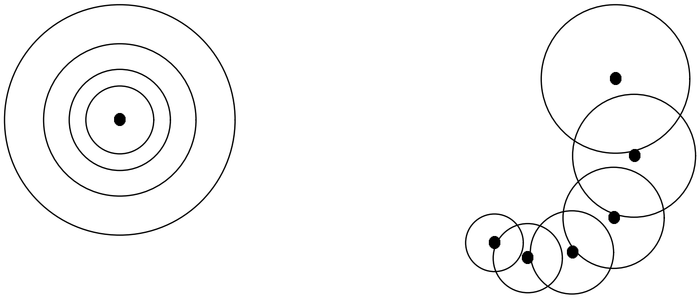
Since the trace is a similarity invariant, we can use $E(z) = Ke^{i\theta}z$ to compute the trace for elliptic, hyperbolic, and loxodromic transformations.

$$\operatorname{Tr} M = \sqrt{Ke^{i\theta}} + \frac{1}{\sqrt{Ke^{i\theta}}}$$

In the elliptic case, $K = 1$ and $\operatorname{Tr} M = 2\cos\frac{\theta}{2}$. For $M(z)$ hyperbolic, $\theta = 0$ and $\operatorname{Tr} M = \sqrt{K} + \frac{1}{\sqrt{K}}$.

We now consider the case in which ∞ is one of the fixed points of $M(z) = \frac{ax+b}{cz+d}$. As mentioned earlier, this corresponds to $c = 0$. If $M(z)$ is parabolic, then $a = d = 1, b = \alpha$ and the conjugating function, $g(z)$, is the identity. In the three cases for which there are two fixed points, $g(z)$ is simply the translation of the finite fixed point, p, to the origin, i.e., $g(z) = z-p$. Further, there are no distinguished isometric circles when ∞ is a fixed point. In this case, elliptic and parabolic Möbius transformations are actually Euclidean isometries, rotations and translations, respectively, so that all circles are isometric. Hyperbolic and loxodromic transformations with ∞ as a fixed point have no isometric circles, since all circles undergo a Euclidean dilation or contraction.

(5.1.4.1)



hyperbolic loxodromic

Images of circles under iteration by indicated $M(z)$ with fixed point at ∞.

$PSL(2,\mathbb{R})$ is the group of hyperbolic isometries for the upper half-plane model of $\mathbb{H}^2$. From the previous discussion, we can conclude that all hyperbolic and parabolic elements of $PSL(2,\mathbb{R})$ have their fixed points on the x-axis. Further, each elliptic element of $PSL(2,\mathbb{R})$ has two fixed points that are symmetric in the x-axis. This is so, because, otherwise, the transformations would fail to preserve the upper half-plane. Finally, there are no loxodromic elements in $PSL(2,\mathbb{R})$. In general, loxodromic elements of $PSL(2,\mathbb{C})$ fail to preserve any disc or half-plane in $\mathbb{C}$ and, thus, cannot be an isometry for any model of $\mathbb{H}^2$.

5.2. A partition of the parameter space for $\operatorname{Tr} A$ due to Jørgensen

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{C})$ such that $A(z)$ has finite fixed points, $\{p_1, p_2\}$, with the understanding that, if $A(z)$ is parabolic, then $p_1 = p_2$. Let $G = \{A\}$ be the cyclic group generated by A. Recall that a fundamental region for G can be constructed by taking the intersection of the exterior of isometric circles for all elements in G. In the present case the fundamental region will be a polygon, which we denote by P. Under the action of G, the polygon P will tile the extended complex plane. In this section, we will denote the isometric circles for A^m as I_m and I_{-m}.

5.2.1. The fundamental polygon P

Consider the fundamental region, P, for a cyclic group of Möbius transformations, G. The side-pairing transformations in G for the fundamental polygon, P, are known to generate G.

As long as $A(z)$, the generator of cyclic group G, is not elliptic, then one of the fixed points is an attractor for all points in the extended complex plane except for the two fixed points. For non-elliptic A, there can be only a finite number of isometric circles of G that lie outside the discs bounded by I_1 and I_{-1}, the isometric circles of the group generator. In particular, there will be only a finite number of isometric circles that bound[19] P. The point at ∞ lies in the interior of P. The fixed points and all isometric circles lie exterior

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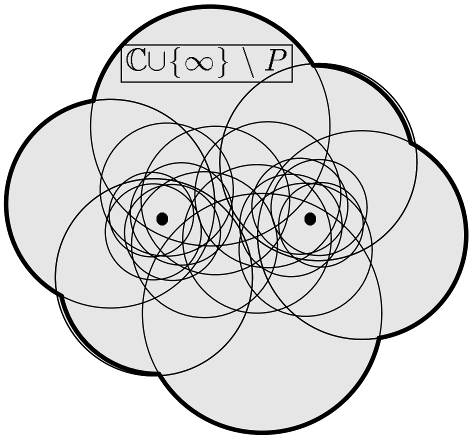
[19] Parabolic transformations generate cyclic groups whose isometric circles *all* intersect the boundary of the fundamental region, but only in a single point.

to P. Which isometric circles bound P will be of great interest to us. The shaded region in the figure below depicts the exterior of P, that is, $\mathbb{C} \cup \{\infty\} \setminus P$.

(5.2.1.1)



$$P = \bigcap_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{C} \cup \{\infty\} \setminus I_m$$

= extended plane minus shaded region.

5.2.2. Isometric circle configurations.

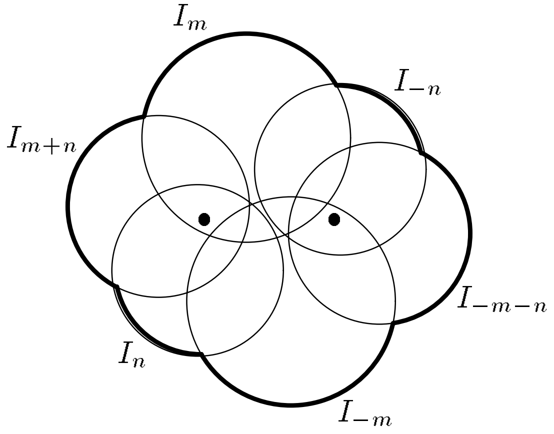
By the *configuration of isometric circles,* we mean the skeleton of the cell complex made up of the isometric circles for all elements in G. The configuration of isometric circles lies outside the interior of P, that is, within the shaded region in the illustration above. Jørgensen proves that the configuration of isometric circles for G is a similarity invariant in $PSL(2, \mathbb{C})$, that is, the relationships of containment and intersection among the isometric circles in the configuration are preserved under similarity transformations. In particular, which isometric circles bound the fundamental polygon, P, will be invariant under similarity transformations in $PSL(2, \mathbb{C})$. Thus, the trace of the group generator determines the type of fundamental polygon, P. Further, the configurations are symmetric with respect to the mid-point of the line segment joining the two fixed points. Thus, without loss of generalization, we may consider only the case where the point of symmetry is the origin and where $\operatorname{Tr} A$ lies in UHP$\cup$\{non-negative x-axis\}.

Jørgensen proves that the fundamental polygon P is bounded by one, two, or three pairs of isometric circles. It is remarkable that, if three pairs of isometric circles bound P, then these isometric circles must be from the group elements A^m, A^{m+n}, and A^n, with m and n relatively prime! The elements whose isometric circles bound the fundamental polygon P will identify pairs of edges of P. Further, these group elements generate the group G. It is for this reason that the powers , m and n, must be relatively prime.

In the case that three pairs of isometric circles bound P, the isometric circles lie in the following order around the boundary of P.

$$I_m I_{m+n} I_n I_{-m} I_{-m-n} I_{-n}$$

(5.2.2.1)



Isometric circles for $A^m, A^n,$ and A^{m+n}.

$$g(z) = \frac{z+1}{z-1},\ A(z) = g^{-1}\left(1.3e^{1.3i}(g(z))\right)$$

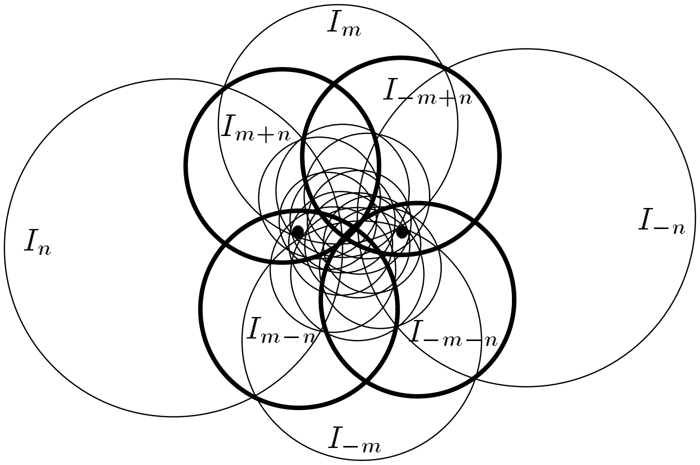
Such a fundamental polygon is said to be of type $\{m, m+n, n\}$.

For the case when there are only two pairs of isometric circles bounding P, the isometric circles are arranged in the order indicated below. Further, the four isometric circles of group elements A^{n+m} and A^{m-n} will intersect, respectively, each of the four vertices of P. The parentheses indicate an isometric circle that intersects the vertex of P but otherwise lies outside of P, that is, off the boundary. In this case, the polygon P is said to be of type $\{m, n\}$.

$$I_m(I_{m+n})I_n(I_{-m+n})I_{-m}(I_{-m-n})I_{-n}(I_{m-n}).$$

In the following figure $A(z) = g^{-1}\left(1.09e^{1.2i}\left(g(z)\right)\right)$, with $g(z) = \frac{z+1}{z-1}$.

(5.2.2.2)

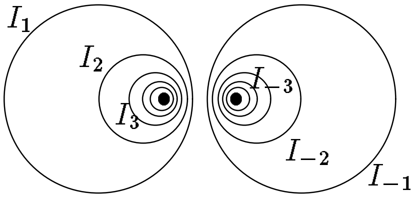


Isometric circles for A^{m+n} and A^{m-n} form triple intersection points.

We now present a tesselation of UHP based on the trace of the cyclic group generator, $\operatorname{Tr} A$, and the type of fundamental polygon for $G = \{A\}$. Elliptic Möbius transformations have real trace values in the half open interval, $[0, 2)$. The fundamental polygon, P, which is associated with elliptic A, will have non-empty interior if and only if $\operatorname{Tr} A = 2\cos q\pi$, with q rational. That is to say, when G is of finite order.

Hyperbolic Möbius transformations have trace values $\operatorname{Tr} A > 2$. The isometric circles will be nested and the fundamental polygon will be doubly connected.

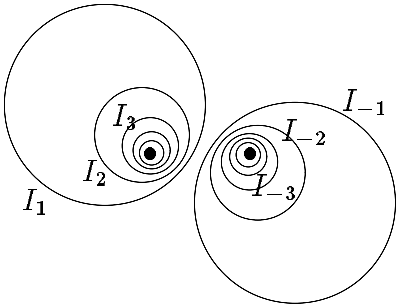
(5.2.2.3)



$\operatorname{Tr} A > 2$, $A(z)$ is hyperbolic.

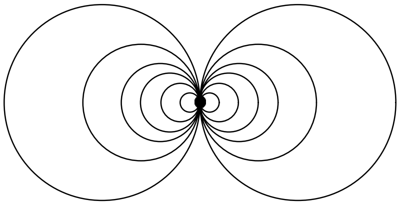
In general, when trace value, $|\operatorname{Tr} A| > 2$, the isometric circles will be nested and the fundamental polygon will be doubly connected.

(5.2.2.4)



$|\operatorname{Tr} A| >= 2, \operatorname{Im}(\operatorname{Tr} A) \neq 0.$

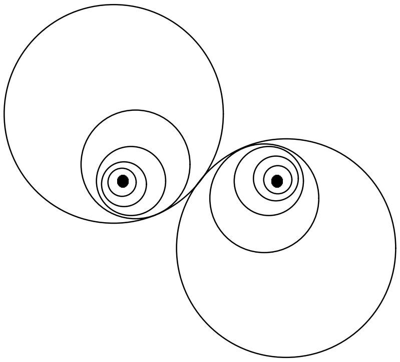
(5.2.2.5)



Tr $A = 2$, $A(z)$ is parabolic.

Below is the isometric circle configuration for loxodromic $A(z)$ with $|\operatorname{Tr} A| = 2$.

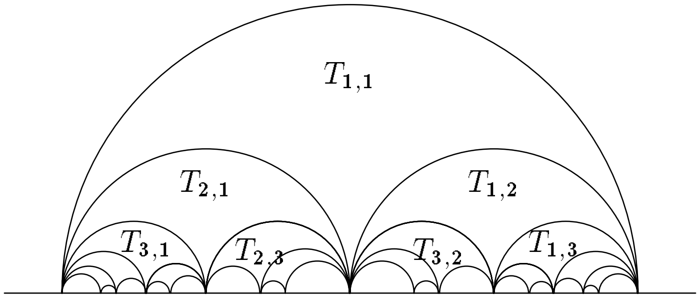
(5.2.2.6)



$|\operatorname{Tr} A| = 2, \operatorname{Tr} A$ not real.

The most interesting cases occur when $|\operatorname{Tr} A| < 2$. Following Jorgensen, we let $f_{m,n}$ denote the locus of points in the upper half-plane for which the corresponding P is of type $\{m, n\}$. Let $T_{m,n}$ denote the locus of points for which P is of type $\{m, m+n, n\}$. Jorgensen shows that the regions $T_{m,n}$ are connected and simply connected with three "vertices," the common end points of $f_{m,n}$, $f_{m,m+n}$, and $f_{m+n,n}$.

(5.2.2.7)



Partition of $\operatorname{Tr} A$ space for $A \in PSL(2,\mathbb{C})$.

Jørgensen proves that $f_{m,n}$ are smooth curves meeting the x-axis orthogonally at the trace values for elliptic transformations of order m and n, respectively. Jørgensen makes no further comment on the possible shapes of $f_{m,n}$. However, we have computational evidence that the arcs are *not* circular.

The Farey fractions are defined iteratively as follows. If $\dfrac{k}{m}$ and $\dfrac{l}{n}$ are two consecutive Farey fractions in the j^{th} step and $m+n \leq j+1$, then $\dfrac{k+l}{m+n}$ is a Farey fraction lying between $\dfrac{k}{m}$ and $\dfrac{l}{n}$ in the $j+1^{st}$ step. The first several steps are written out below to illustrate the procedure.

$$\begin{array}{ccccccccccc}
\frac{0}{1} & & & & & & & & & & \frac{1}{1} \\[2ex]
\frac{0}{1} & & & & & \frac{1}{2} & & & & & \frac{1}{1} \\[2ex]
\frac{0}{1} & & & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & & & \frac{1}{1} \\[2ex]
\frac{0}{1} & & \frac{1}{4} & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & \frac{3}{4} & & \frac{1}{1} \\[2ex]
\frac{0}{1} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{1}{1}
\end{array}$$

First five stages of Farey fraction production.

As a set, Farey fractions are simply all rational numbers in the unit interval. There will be a region $T_{m,n}$ in the partition of the trace parameter space if and only if there is a pair of consecutive fractions $\dfrac{k}{m}$ and $\dfrac{l}{n}$ at some step in the procedure described above. In this case, an element of $PSL(2,\mathbb{C}) : A(z) = \frac{az+b}{cz+d}, ad - bc = 1,$ such that $a + d \in T_{m,n}$ will generate a cyclic loxodromic group with isometric circle configuration bounded by the isometric circles of $A^m, A^{m+n},$ and A^n.

Every elliptic element of $PSL(2,\mathbb{C})$ has trace $= 2\cos\frac{\theta}{2}, 0 < \theta < 2\pi$. Every elliptic element with finite order will have $\theta = \frac{k}{m}2\pi$. Jørgensen proves that within the triangular region, $T_{m,n}$ (which lies within the disc of radius 2 and with its vertices lying on the x-axis at $2\cos\frac{k}{m}\pi, 2\cos\frac{k+l}{m+n}\pi,$ and $2\cos\frac{l}{n}\pi$,) the six edges of P will lie on the isometric circles of $A^m, A^n,$ and A^{m+n}. Möbius transformations with traces equal to these values on the vertices (which lie on the x-axis,) are elliptic elements of $PSL(2,\mathbb{C})$ with finite orders m, $\frac{m+n}{(m+n,k+l)}$, and n, respectively.

Jørgensen proves that the only way a new side can emerge for P, is for an isometric circle to break through an existing vertex of P. That is to say, as Tr A varies and the fundamental polygon varies in shape, an isometric circle can break out through the boundary of the configuration only through existing vertices of P. A triple intersection point occurs precisely when the circle breaks through the vertex. In particular, a new side is never formed by an isometric circle breaking through an edge, it must be a through vertex of P.

When P is bounded by just two pairs of isometric circles, each of the four vertices is a triple intersection point of isometric circles. If A^m and A^n are the group elements corresponding to the isometric circles bounding P, then there will be two pairs of isometric circles that pass through the vertices, forming triple intersection points together with the edges of P, but otherwise lying outside of P. These will be the isometric circles for the elements A^{m+n} and A^{m-n}. A fundamental polygon of type $\{m, m+n, n\}$ will be produced if $\operatorname{Tr} A$ is perturbed in one direction and of type $\{m-n, m, n\}$ if $\operatorname{Tr} A$ is perturbed in the other direction. Thus, a fundamental polygon with four sides is a transition between two related types of six-sided fundamental polygons.

5.2.3. Fundamental polyhedra in $\mathbb{H}^3$.

Suppose we denote by S_k the solid upper-hemisphere which has as its equator the isometric circle, I_k. Further, assume the Poincaré (hyperbolic) metric in the upper half-space (UHS.) Upper semi-circles orthogonal to the xy-plane are geodesics. Via the Poincaré extension, we can extend the action of G into all of the upper half-space. The fundamental polygon P, becomes just one of at least three faces of a fundamental polyhedron, P_h, which we define in a completely analogous manner to the definition of the fundamental polygon, P.

$$P_h = \bigcap_{A^k \in G} UHS \setminus S_k \tag{5.2.3.1}$$

The fundamental polygon P, is the face of the polyhedron lying in the boundary plane of UHS. All other bounding faces lie on the hemispheres of group elements. Any sphere S_K will lie interior to sphere S_L if and only if the isometric circle I_K lies within I_L. Thus, a circle, I_K, which is exterior to P but near a vertex of P may well have its hemisphere, S_K, exposed above the two intersecting spheres whose equators bound P in the plane. The exposed portion of an isometric sphere which does not contact the plane on the boundary of P is called a *facet* and the portions which do contact the plane and, therefore, bound P are referred to as *sides* of the polyhedron P_h.

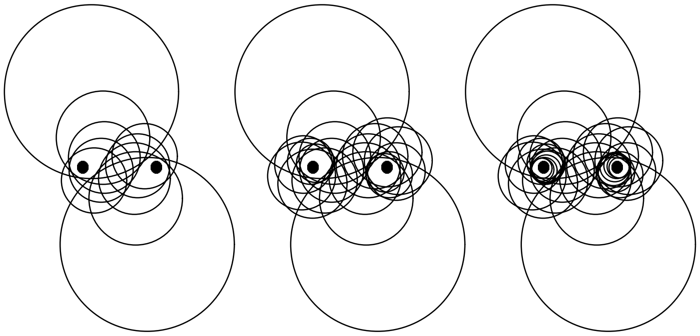
The configuration of hemispheres will have one, two, or three pairs of sides, that is, hemispheres that contact the plane on the boundary of P. Other isometric hemispheres may bound the polyhedron P_h. These hemispheres must do so up above the xy-plane. We refer to the crease-shaped fold where two hemispheres intersect as their *pleat*. It is in the pleat of two intersecting hemispheres that an isometric hemisphere, S_k, can be exposed to bound P_h even when the isometric circle, I_k, the equator of S_k, lies exterior to P. For a fundamental polygon P of type $\{m, n\}$, the isometric circles $I_{m+n}, I_{m-n}, I_{-m-n}$, and I_{-m+n} form the triple intersection points at the vertices of P. The hemispheres $S_{m+n}, S_{m-n}, S_{-m-n}$, and S_{-m+n} will meet the polygon P at only these respective vertices. However the hemisphere S_{m+n}, for instance, bounds the polyhedron P_h along a lens-shaped region of S_{m+n} that lies in the pleat between S_m and S_n.

Color images of the hyperbolic polyhedra and of related geometric objects can be found at the web address http://new.math.uiuc.edu/~paulmcc/.

When the isometric circle recedes within the isometric circle configuration and away from the vertex of P, then S_{m+n} will have its exposed facet lift above the xy-plane. There can be an arbitrarily large number of isometric hemispheres with facets exposed and bounding the polyhedron P_h.

The series of figures below show the isometric circles of the first 5, 10, and then 15 powers of the loxodromic transformation $A(z) = g^{-1} \circ E \circ g(z)$, with $E(z) = 1.3e^{i.825}z$ and $g(z) = \frac{z+1}{z-1}$.

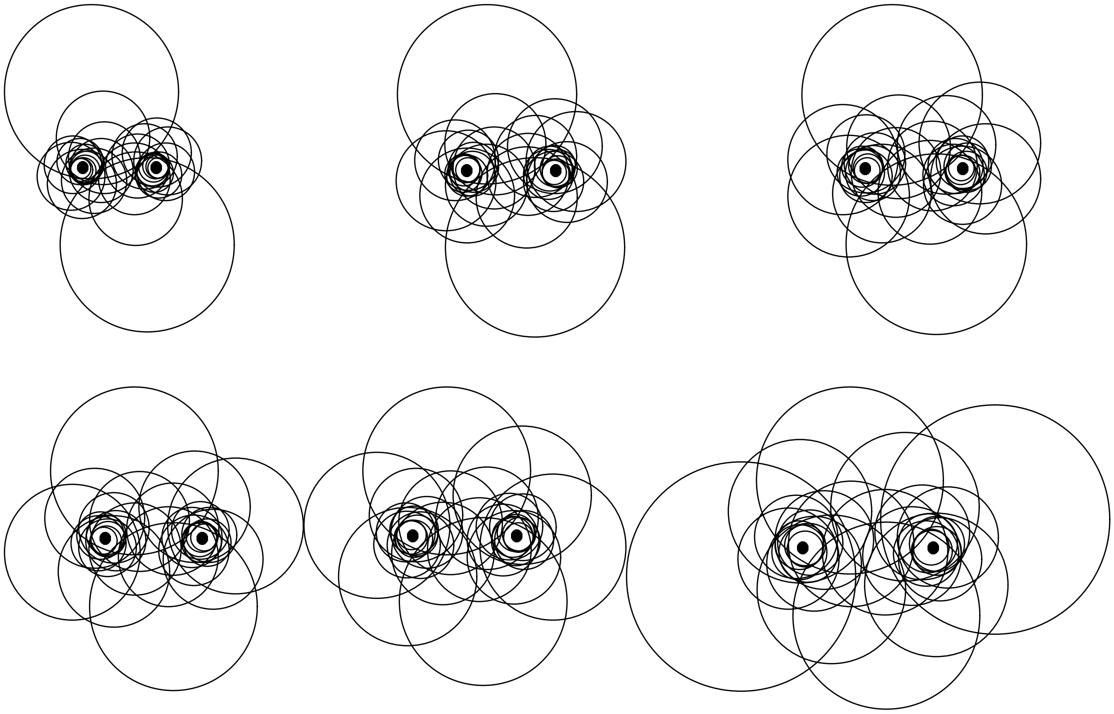
(5.2.2)



Isometric circle configurations: first 5, 10, and 15 powers of the group generator.

Below are the isometric circle configurations for the cyclic groups with generators $A(z) = g^{-1} \circ E \circ g(z)$, with $E(z) = 1.3e^{i\theta}z, g(z) = \frac{z+1}{z-1}$, and with θ ranging from .825 radians to 1.5 radians. Note that when θ is approximately 1.2 radians, the type of fundamental polygon changes; triple intersection points form as an isometric circle pushes through the lower left vertex of the fundamental polygon in the first figure of the second row. Notice also that the relative sizes of the isometric circles change as the parameters vary.

(5.2.3)



$K = 1.3$ $\theta = .825, 9.9, 1.1, 1.2, 1.3, 1.5$